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On generalized 2-step continuous linear multistep method of hybrid type for the integration of second order ordinary differential equations

[1]J. O. Ehigie, [2]S. A. Okunuga, [3]A. B. Sofoluwe, [4]M. A. Akanbi

[1,2]Department of Mathematics, University of Lagos, Akoka, Lagos
[3]Department of Computer Science, University of Lagos, Akoka, Lagos
[4]Department of Mathematics, Lagos State University, Ojo. Lagos

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ABSTRACT:

This paper proposes a generalized 2-step continuous multistep method of hybrid type for the direct integration of second-order ordinary differential equations in a multistep collocation technique, which yields block methods. The scheme obtained is used as a single continuous form which serves as a family of formula involving (x,s) such that on substitution of an off-step point s a bi-hybrid continuous scheme is obtained. The discrete equivalent is also obtained thereafter from the continuous family of formula as a block method. It was discovered that Numerical schemes of StÖrmer-Cowell type were recovered via this technique. The scheme obtained is implemented to generate the numerical solution to second order ordinary differential equations. The results obtained are compared with the Renowned Numerov method, known to be of optimal

Keywords: Collocation, Block method, hybrid, StÖrmer-Cowell, Numerov, Optimal Order

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INTRODUCTION

Many field of applications, notably in Science and Engineering yields initial value problems of Second order ordinary differential equations are of the form,

$$y''(x) = f(x, y, y'), \quad y(a) = y_0, \quad y'(a) = \beta \tag{1.1}$$

Many of such problems may not be easily solved analytically, hence numerical schemes are developed to solve (1.1). These equations are usually reduced to 2 systems of first order ordinary differential equations and numerical methods of First Order differential equations are used to solve them. Linear Multistep Methods are powerful numerical methods for solving Differential equations.

Some researchers have attempted the solution of (1.1) using linear multistep methods without reduction to systems of first order ordinary differential equations, they include Brown (1977), Lambert (1991), Kayode (2005), Adee et al. (2005), Onumanyi et al. (2008), Yahaya et al (2009) to mention a few. Lambert (1973) also discussed about an optimal two step method called the Numerov's Method.

In our previous paper, we developed generalized schemes of continuous linear multistep methods (CLMS) of hybrid type for solving first order ordinary differential equations, Ehigie et al. (2010). Here, we extend our results to generalizing the solution to Second Order ordinary differential equation via multistep collocation technique. This method helps to provide a continuous numerical scheme which accommodates all hybrid points, so that on substitution of an off-step point s a bi-hybrid scheme is obtained for the direct integration of (1.1) without reduction to systems of differential equations.

In section 2, we present the theoretical procedure used as a generalization for all cases in this class. In section 3, the generalized method is derived as a special case of the theoretical procedure for $m=2$. Examples of schemes generated from the generalized methods were derived in section 4. The Schemes generated were implemented on some problems in section 5. Finally, conclusion and future research areas were also stated.

1. Theoretical Procedure:
2.

Consider the system,

$$y''(x) = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0 \tag{2.1}$$

The Numerical Solution to (2.1) can be obtained using a $m-$step explicit Linear Multistep Method (LMM) of the form,

$$\sum_{j=0}^{m} \alpha_j y_{k+j} = h^2 \sum_{j=0}^{m-1} \beta_j f_{k+j} \tag{2.2}$$

for a second-order differential equation, Lambert (1973), but since we are interested in obtaining an hybrid scheme, 2 hybrid points, $(1-s)$ and $(1+s)$ were introduced as a generalization of all hybrid methods of this class so that (2.2) becomes,

$$\sum_{j=0}^{m} \alpha_j y_{k+j} = h^2 \sum_{j=0}^{m-1} \beta_j f_{k+j} + h^2 \beta_{s+1} f_{k+s+1} + h^2 \beta_{1-s} f_{k-s+1} \tag{2.3}$$

To obtain the CLMS for (2.1), it is necessary to use the power series,

$$Y(x) = \sum_{j=0}^{2m} a_j \left(\frac{x - x_k}{h} \right)^j \tag{2.4}$$

where m is the $m-$step method of this class.

Let m and n be numbers of interpolations and collocations respectively, then we shall have a system of $(2m+1)$ equations of the form represented in the matrix below,

$$\begin{pmatrix} d_{11} & d_{12} & d_{13} & \cdots & d_{1m} & d_{1m+1} & d_{1m+2} & \cdots & d_{12m+1} \\ d_{21} & d_{22} & d_{23} & \cdots & d_{2m} & d_{2m+1} & d_{2m+2} & \cdots & d_{22m+1} \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & d_{m3} & \cdots & d_{mm} & d_{mm+1} & d_{mm+2} & \cdots & d_{m2m+1} \\ d_{m+11} & d_{m+12} & d_{m+13} & \cdots & d_{m+1m} & d_{m+1m+1} & d_{m+1m+2} & \cdots & d_{m+12m+1} \\ d_{m+21} & d_{m+22} & d_{m+23} & \cdots & d_{m+2m} & d_{m+2m+1} & d_{m+2m+2} & \cdots & d_{m+22m+1} \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{2m-11} & d_{2m-12} & d_{2m-13} & & d_{2m-1m} & d_{2m-1m+1} & d_{2m-1m+2} & & d_{2m-12m+1} \\ 0 & 0 & f_0(s) & \cdots & f_{m-2}(s) & f_{m-1}(s) & f_m(s) & \cdots & f_{2m}(s) \\ 0 & 0 & g_0(s) & \cdots & g_{m-2}(s) & g_{m-1}(s) & g_m(s) & & g_{2m}(s) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ . \\ . \\ . \\ a_{2m} \end{pmatrix} = \begin{pmatrix} Y_k \\ Y_{k+1} \\ \vdots \\ Y_{k+m-1} \\ h^2 f_{k+1} \\ \vdots \\ h^2 f_{k+m-1} \\ h^2 f_{k+s+1} \\ h^2 f_{k-s+1} \end{pmatrix} \quad (2.5)$$

Where d_{ij} are as derived in the interpolations and collocations respectively as derived in Okunuga et al (2009) and $f_p(s)$ and $g_p(s)$ are polynomials of degree p. i.e,

$$f_p(s) = \sum_{i=0}^{p} d_i s^i$$

and,

$$g_p(s) = \sum_{i=0}^{p} d_i s^i$$

Expressing (2.5) further as,

$$DA = C$$

$$\Rightarrow \qquad A = D^{-1}C$$

where $A \Rightarrow a_j = \varphi(s, h, Y_{k+j}, f_{k+j}, f_{k-s+1}, f_{k+s+1})$, For $j = 0,1,2,\cdots,2m$

substituting: a_j, $j = 0,1,2,\cdots,2m$ in (2.4), we shall obtain,

$$Y(x,s) = \sum^{m-1} \alpha_j(x,s) Y_{k+j} + h^2 \sum^{m-1} \beta_j(x,s) f_{k+j} + h^2 \beta_{s+1}(x,s) f_{k+s+1} + h^2 \beta_{1-s}(x,s) f_{k-s+1}$$

such that for any appropriate value of s, a new CLMS of hybrid type is generated and given as,

$$Y(x) = \sum^{m-1} \alpha_j(x) Y_{k+j} + h^2 \sum^{m-1} \beta_j(x) f_{k+j} + h^2 \beta_{s+1}(x) f_{k+s+1} + h^2 \beta_{1-s}(x) f_{k-s+1}$$

where, $\alpha_j(x) = \sum_{i=0}^{2m} \alpha_{ji} \left(\frac{x - x_k}{h} \right)^i$ and $\beta_j(x) = \sum_{i=0}^{2m} \beta_{ji} \left(\frac{x - x_k}{h} \right)^i$

This can be used to generate block methods on evaluation of $Y(x)$ at $x = x_{k+m}, x_{k-s+1}$ and x_{k+s+1} respectively.

3. Derivation of the Generalized method for *m = 2*

For such a $2-step$ hybrid method, i.e. setting $m=2$. Let the basis polynomial to the problem (2.1) be given as,

$$Y(x)=\sum_{j=0}^{2m} a_j\left(\frac{x-x_k}{h}\right)^j \tag{3.1}$$

Interpolating $x_j : j=0,1$ and also collocating $Y''(x)$ at $x_j : j=0,1,(1-s)$ and $(1+s)$
We shall obtain the system of equation express by the matrix,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & (6+6s) & (12+24s+12s^2) \\ 0 & 0 & 2 & 6 & 12 \\ 0 & 0 & 2 & (6-6s) & (12-24s+12s^2) \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}=\begin{pmatrix} Y_k \\ Y_{k+1} \\ f_{k+s+1} \\ f_{k+1} \\ f_{k-s+1} \end{pmatrix} \tag{3.2}$$

Solving this system of equations with maple software. The hybrid term s was delayed so as to obtain a generalized schemes rather than an individual continuous formula which is limited to a particular s values as in Yahaya et al. (2009). We obtain,

$$a_0 = Y_k$$

$$a_1 = -\frac{1}{24s^2}\left(24Y_k s^2 + 3h^2 f_{k+s+1} - 6h^2 f_{k+1} + 3h^2 f_{k-s+1} - 4h^2 s f_{k+s+1} + 4h^2 f_{k-s+1}s - 24Y_{k+1}s^2 + 12h^2 f_{k+1}s^2\right)$$

$$a_2 = \frac{h^2}{4s^2}\left(f_{k+s+1} - 2f_{k+1} + f_{k-s+1} - s f_{k+s+1} + f_{k-s+1}s + 2f_{k+1}s^2\right)$$

$$a_3 = -\frac{h^2}{12s^2}\left(2f_{k+s+1} - 4f_{k+1} + 2f_{k-s+1} - s f_{k+s+1} + f_{k-s+1}s\right)$$

$$a_4 = \frac{h^2}{24}\left(f_{k+s+1} - 2f_{k+1} + f_{k-s+1}\right)$$

Substituting a_j, $j=0, 1, 2, 3$ and 4 into (3.1), we obtain,

$$Y(x,s)=$$

$$\left[1-\left(\frac{x-x_k}{h}\right)\right]Y_k + \left(\frac{x-x_k}{h}\right)Y_{k+1} + h^2\left[\begin{array}{l}\left(\frac{1}{6s}-\frac{1}{8s^2}\right)\left(\frac{x-x_k}{h}\right)+\left(\frac{1}{4s^2}-\frac{1}{4s}\right)\left(\frac{x-x_k}{h}\right)^2 \\ +\left(\frac{1}{12s}-\frac{1}{6s^2}\right)\left(\frac{x-x_k}{h}\right)^3+\frac{1}{24s^2}\left(\frac{x-x_k}{h}\right)^4\end{array}\right]f_{k+s+1}$$

$$+h^2\left[\begin{array}{l}\left(\frac{1}{4s^2}-\frac{1}{2}\right)\left(\frac{x-x_k}{h}\right)+\left(\frac{1}{2}-\frac{1}{2s^2}\right)\left(\frac{x-x_k}{h}\right)^2 \\ +\frac{1}{3s^2}\left(\frac{x-x_k}{h}\right)^3-\frac{1}{12s^2}\left(\frac{x-x_k}{h}\right)^4\end{array}\right]f_{k+1}$$

$$+h^2\begin{bmatrix}-\left(\frac{1}{6s}+\frac{1}{8s^2}\right)\left(\frac{x-x_k}{h}\right)+\left(\frac{1}{4s}+\frac{1}{4s^2}\right)\left(\frac{x-x_k}{h}\right)^2\\-\left(\frac{1}{12s}+\frac{1}{6s^2}\right)\left(\frac{x-x_k}{h}\right)^3+\frac{1}{24s^2}\left(\frac{x-x_k}{h}\right)^4\end{bmatrix}f_{k-s+1} \quad .(3.3)$$

and its first derivative is,

$Y'(x,s)=$

$$\left[-\frac{1}{h}\right]Y_k+\left(\frac{1}{h}\right)Y_{k+1}+h\begin{bmatrix}\left(\frac{1}{6s}-\frac{1}{8s^2}\right)+\left(\frac{1}{2s^2}-\frac{1}{2s}\right)\left(\frac{x-x_k}{h}\right)+\left(\frac{1}{4s}-\frac{1}{2s^2}\right)\left(\frac{x-x_k}{h}\right)^2\\+\frac{1}{6s^2}\left(\frac{x-x_k}{h}\right)^3\end{bmatrix}f_{k+s+1}$$

$$+h\begin{bmatrix}\left(\frac{1}{4s^2}-\frac{1}{2}\right)+\left(1-\frac{1}{s^2}\right)\left(\frac{x-x_k}{h}\right)\\+\frac{1}{s^2}\left(\frac{x-x_k}{h}\right)^2-\frac{1}{3s^2}\left(\frac{x-x_k}{h}\right)^3\end{bmatrix}f_{k+1} \qquad (3.4)$$

$$+h\begin{bmatrix}-\left(\frac{1}{6s}+\frac{1}{8s^2}\right)+\left(\frac{1}{2s}+\frac{1}{2s^2}\right)\left(\frac{x-x_k}{h}\right)-\\\left(\frac{1}{4s}+\frac{1}{2s^2}\right)\left(\frac{x-x_k}{h}\right)^2+\frac{1}{6s^2}\left(\frac{x-x_k}{h}\right)^3\end{bmatrix}f_{k-s+1}$$

$Y(x,s)$ is evaluated at $x=x_{k-s+1}$, x_{k+s+1} and x_{k+m} respectively to obtain the generalized discrete hybrid schemes from (3.3) as,

$$Y_{k-s+1}-[1-s]Y_{k+1}-sY_k=$$
$$h^2\left[\left(\frac{1}{12}-\frac{1}{24s}-\frac{s^2}{24}\right)f_{k+s+1}+\left(-\frac{s}{2}+\frac{1}{12s}+\frac{5s^2}{12}\right)f_{k+1}+\left(-\frac{1}{12}-\frac{1}{24s}+\frac{s^2}{8}\right)f_{k-s+1}\right]$$
$$Y_{k+s+1}-[1+s]Y_{k+1}+sY_k=$$
$$h^2\left[\left(-\frac{1}{12}+\frac{1}{24s}+\frac{s^2}{8}\right)f_{k+s+1}+\left(\frac{s}{2}-\frac{1}{12s}+\frac{5s^2}{12}\right)f_{k+1}+\left(\frac{1}{12}+\frac{1}{24s}-\frac{s^2}{24}\right)f_{k+s+1}\right]$$
$$Y_{k+2}-2Y_{k+1}+Y_k=h^2\left[\frac{1}{12s^2}f_{k+s+1}+\left(1-\frac{1}{6s^2}\right)f_{k+1}+\left(\frac{1}{12s^2}\right)f_{k-s+1}\right] \qquad (3.5)$$

(3.3) which involves function of (x,s) is the generalized continuous formulation of a class of $2-$step continuous hybrid scheme which yield the generalized discrete block methods (3.5) on substitution of an appropriate value of $s\in Q, s\in(0,1)$. Similarly, the derivative (3.4) is evaluated at $x=x_{k-s+1}\ x_{k+s+1}$, and x_{k+m} respectively to obtain a generalized block scheme,

$$Y'_{k-s+1}-\frac{1}{h}Y_{k+1}+\frac{1}{h}Y_k=$$
$$h\left[\left(\frac{s}{12}-\frac{1}{12s}+\frac{1}{24s^2}\right)f_{k+s+1}+\left(\frac{1}{2}-\frac{2s}{3}-\frac{1}{12s^2}\right)f_{k+1}+\left(\frac{1}{12s}+\frac{1}{24s^2}-\frac{5s}{12}\right)f_{k-s+1}\right]$$
$$Y'_{k+s+1}-\frac{1}{h}Y_{k+1}+\frac{1}{h}Y_k=$$
$$h\left[\left(-\frac{1}{12s}+\frac{5s}{12}+\frac{1}{24s^2}\right)f_{k+s+1}+\left(\frac{1}{2}-\frac{1}{12s^2}+\frac{2s}{3}\right)f_{k+1}+\left(\frac{1}{12s}+\frac{1}{24s^2}-\frac{s}{12}\right)f_{k-s+1}\right]$$
$$Y'_{k+2}-\frac{1}{h}Y_{k+1}+\frac{1}{h}Y_k=h\left[\left(\frac{5}{24s^2}+\frac{1}{6s}\right)f_{k+s+1}+\left(\frac{3}{2}-\frac{5}{12s^2}\right)f_{k+1}+\left(\frac{5}{24s^2}-\frac{1}{6s}\right)f_{k-s+1}\right] \quad (3.6)$$

which is used to obtain derivative values (y') present in the $f_{n+j}=f(x_{n+j},y_{n+j},y'_{n+j})$ in the main problem (2.1).

4. Examples of schemes derived from the generalized continuous scheme

We shall now substitute into the generalized CLMS (3.3) at $s=\frac{1}{4}$, we obtain the CLMS,

$$Y(x)=\left(\frac{x-x_k}{h}\right)Y_{k+1}+\left[1-\left(\frac{x-x_k}{h}\right)\right]Y_k+$$
$$h^2\left[\begin{array}{l}\left[-\frac{4}{3}\left(\frac{x-x_k}{h}\right)+3\left(\frac{x-x_k}{h}\right)^2-\frac{7}{3}\left(\frac{x-x_k}{h}\right)^3+\frac{2}{3}\left(\frac{x-x_k}{h}\right)^4\right]f_{k+\frac{5}{4}}\\+\left[\frac{7}{2}\left(\frac{x-x_k}{h}\right)-\frac{15}{2}\left(\frac{x-x_k}{h}\right)^2+\frac{16}{3}\left(\frac{x-x_k}{h}\right)^3-\frac{4}{3}\left(\frac{x-x_k}{h}\right)^4\right]f_{k+1}\\+\left[-\frac{8}{3}\left(\frac{x-x_k}{h}\right)+5\left(\frac{x-x_k}{h}\right)^2-3\left(\frac{x-x_k}{h}\right)^3+\frac{2}{3}\left(\frac{x-x_k}{h}\right)^4\right]f_{k+\frac{3}{4}}\end{array}\right] \quad (4.1a)$$

Substituting $s=\frac{1}{4}$ in the generalized discrete block scheme or evaluating (4.1a) at $x=x_{k+2},x_{k+s+1}$ and x_{k-s+1} yields,

$$Y_{k+\frac{3}{4}}-\frac{3}{4}Y_{k+1}-\frac{1}{4}Y_k=\frac{h^2}{128}\left[-31f_{k+\frac{3}{4}}+30f_{k+1}-11f_{k+\frac{5}{4}}\right]$$
$$Y_{k+\frac{5}{4}}-\frac{5}{4}Y_{k+1}+\frac{1}{4}Y_k=\frac{h^2}{384}\left[95f_{k+\frac{3}{4}}-70f_{k+1}+35f_{k+\frac{5}{4}}\right]$$
$$Y_{k+2}-2Y_{k+1}+Y_k=\frac{h^2}{3}\left[4f_{k+\frac{3}{4}}-5f_{k+1}+4f_{k+\frac{5}{4}}\right] \quad (4.1b)$$

Substituting again into the generalized CLMS (3.3) at $s=\frac{1}{2}$, we obtain the CLMS,

$$Y(x)=\left(\frac{x-x_k}{h}\right)Y_{k+1}+\left[1-\left(\frac{x-x_k}{h}\right)\right]Y_k+h^2\left[\begin{array}{l}\left[-\frac{1}{6}\left(\frac{x-x_k}{h}\right)+\frac{1}{2}\left(\frac{x-x_k}{h}\right)^2-\frac{1}{2}\left(\frac{x-x_k}{h}\right)^3+\frac{1}{6}\left(\frac{x-x_k}{h}\right)^4\right]f_{k+\frac{3}{2}}\\+\left[\frac{1}{2}\left(\frac{x-x_k}{h}\right)-\frac{3}{2}\left(\frac{x-x_k}{h}\right)^2+\frac{4}{3}\left(\frac{x-x_k}{h}\right)^3-\frac{1}{3}\left(\frac{x-x_k}{h}\right)^4\right]f_{k+1}\\+\left[-\frac{5}{6}\left(\frac{x-x_k}{h}\right)+\frac{3}{2}\left(\frac{x-x_k}{h}\right)^2-\frac{5}{6}\left(\frac{x-x_k}{h}\right)^3+\frac{1}{6}\left(\frac{x-x_k}{h}\right)^4\right]f_{k+\frac{1}{2}}\end{array}\right] \quad (4.2a)$$

Substituting $s=\frac{1}{2}$ in the generalized discrete block scheme or evaluating (4.2a) at $x=x_{k+2}, x_{k+s+1}$ and x_{k-s+1} yields,

$$\begin{aligned}Y_{k+\frac{1}{2}}-\frac{1}{2}Y_{k+1}-\frac{1}{2}Y_k&=\frac{h^2}{96}\left[-13f_{k+\frac{1}{2}}+2f_{k+1}-f_{k+\frac{3}{2}}\right]\\Y_{k+\frac{3}{2}}-\frac{3}{2}Y_{k+1}+\frac{1}{2}Y_k&=\frac{h^2}{32}\left[5f_{k+\frac{1}{2}}+6f_{k+1}+f_{k+\frac{3}{2}}\right]\\Y_{k+2}-2Y_{k+1}+Y_k&=\frac{h^2}{3}\left[f_{k+\frac{1}{2}}+f_{k+1}+f_{k+\frac{3}{2}}\right]\end{aligned} \quad (4.2b)$$

which are the schemes derived in Yahaya et al (2009).

Also substituting (3.3) at $s=\frac{3}{4}$, we obtain the CLMS,

$$Y(x)=\left(\frac{x-x_k}{h}\right)Y_{k+1}+\left[1-\left(\frac{x-x_k}{h}\right)\right]Y_k+$$

$$h^2\left[\begin{array}{l}\left[\frac{1}{9}\left(\frac{x-x_k}{h}\right)^2-\frac{5}{27}\left(\frac{x-x_k}{h}\right)^3+\frac{2}{27}\left(\frac{x-x_k}{h}\right)^4\right]f_{k+\frac{7}{4}}\\+\left[-\frac{1}{18}\left(\frac{x-x_k}{h}\right)-\frac{7}{18}\left(\frac{x-x_k}{h}\right)^2+\frac{16}{27}\left(\frac{x-x_k}{h}\right)^3-\frac{4}{27}\left(\frac{x-x_k}{h}\right)^4\right]f_{k+1}\\+\left[-\frac{4}{9}\left(\frac{x-x_k}{h}\right)+\frac{7}{9}\left(\frac{x-x_k}{h}\right)^2-\frac{11}{27}\left(\frac{x-x_k}{h}\right)^3+\frac{2}{27}\left(\frac{x-x_k}{h}\right)^4\right]f_{k+\frac{1}{4}}\end{array}\right] \quad (4.3a)$$

Substituting $s=\frac{3}{4}$ in the generalized discrete block scheme or evaluating (4.3a) at $x=x_{k+2}, x_{k+s+1}$ and x_{k-s+1} yields,

$$\begin{aligned}Y_{k+\frac{1}{4}}-\frac{1}{4}Y_{k+1}-\frac{3}{4}Y_k&=\frac{h^2}{1152}\left[-79f_{k+\frac{1}{4}}-34f_{k+1}+5f_{k+\frac{7}{4}}\right]\\Y_{k+\frac{7}{4}}-\frac{7}{4}Y_{k+1}+\frac{3}{4}Y_k&=\frac{h^2}{1152}\left[133f_{k+\frac{1}{4}}+574f_{k+1}+49f_{k+\frac{7}{4}}\right]\\Y_{k+2}-2Y_{k+1}+Y_k&=\frac{h^2}{27}\left[4f_{k+\frac{1}{4}}+19f_{k+1}+4f_{k+\frac{7}{4}}\right]\end{aligned} \quad (4.3b)$$

5. Implementation of the Methods

To be able to implement these block schemes (4.1), (4.2) and (4.3). Many Explicit schemes were generated here in abundance via multistep collocation technique to evaluate starting values. These schemes were generated to predict some of the values expected in our schemes. But these schemes are used with respect to the s in use.

Although, Taylor's series was used to evaluate the values for y_{k+1}. i.e,

$$y_{k+1} = y_k + hy'_k + \frac{h^2}{2!} f_k + \frac{h^3}{3!}\left\{\frac{\partial f_k}{\partial x_k} + y'_k \frac{\partial f_k}{\partial y_k} + f_k \frac{\partial f_k}{\partial y'_k}\right\} + O(h^4)$$

and,
$$y'_{k+1} = \frac{1}{h}\left(y_{k+1} - y_k\right) + \frac{1}{2} hf_k$$

for y'_{k+1}. Note that that y_k and y'_k are the initial values given in the problem (2.1).

Numerical Examples

Four examples are solved to demonstrate our derived methods for values of s, $y(s)$ being the numerical solution at value s. Our results from block methods (4.1), (4.2), (4.3) are compared with the Numerov method which is of optimal order.

1. $y'' + y = 0$, $y(0) = 1 = y'(0)$, $h = 0.1$
   Analytical Solution: $y(x) = Cosx + Sinx$

2. $y'' - 100y = 0$, $y(0) = 1$, $y'(0) = -10$, $h = 0.01$
   Analytical Solution: $y(x) = e^{-10x}$

3. $y'' - y' = 0$, $y(0) = 0$, $y'(0) = -1$, $h = 0.1$
   Analytical Solution: $y(x) = 1 - e^{x}$

4. $y'' = \frac{(y')^2}{2y} - 2y$, $y\left(\frac{\pi}{6}\right) = \frac{1}{4}$, $y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ $h = 0.01$
   Analytical Solution: $y(x) = Sin^2 x$

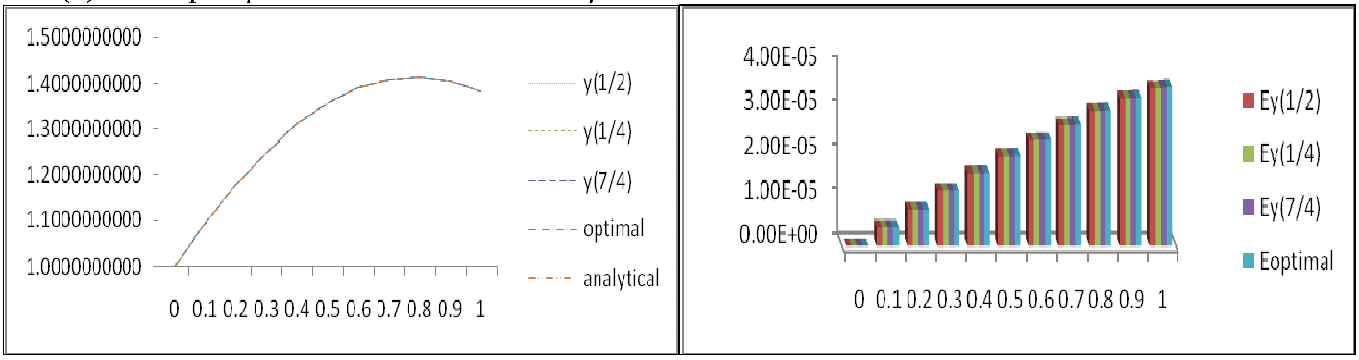
RESULTS

1(a) *Numerical Solution and Table of Errors*

x	y(1/2)	y(1/4)	y(7/4)	optimal	analytical
0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	1.0948333333	1.0948333333	1.0948333333	1.0948333333	1.0948375819
0.2	1.1787274551	1.1787274552	1.1787274527	1.1787274493	1.1787359086
0.3	1.2508441229	1.2508441236	1.2508441157	1.2508441054	1.2508566958
0.4	1.3104627711	1.3104627729	1.3104627564	1.3104627355	1.3104793363
0.5	1.3569877099	1.3569877136	1.3569876850	1.3569876496	1.3570081005
0.6	1.3899540774	1.3899540842	1.3899540396	1.3899539860	1.3899780883
0.7	1.4090324847	1.4090324959	1.4090324314	1.4090323559	1.4090598745
0.8	1.4140323067	1.4140323236	1.4140322353	1.4140321344	1.4140628002
0.9	1.4049035867	1.4049036110	1.4049034950	1.4049033653	1.4049368779
1	1.3817375360	1.3817375694	1.3817374217	1.3817372603	1.3817732907

x	Ey(1/2)	Ey(1/4)	Ey(7/4)	Eoptimal
0	0.00E+00	0.00E+00	0.00E+00	0.00E+00
0.1	4.25E-06	4.25E-06	4.25E-06	4.25E-06
0.2	8.45E-06	8.45E-06	8.46E-06	8.46E-06
0.3	1.26E-05	1.26E-05	1.26E-05	1.26E-05
0.4	1.66E-05	1.66E-05	1.66E-05	1.66E-05
0.5	2.04E-05	2.04E-05	2.04E-05	2.05E-05
0.6	2.40E-05	2.40E-05	2.40E-05	2.41E-05
0.7	2.74E-05	2.74E-05	2.74E-05	2.75E-05
0.8	3.05E-05	3.05E-05	3.06E-05	3.07E-05
0.9	3.33E-05	3.33E-05	3.34E-05	3.35E-05
1	3.58E-05	3.57E-05	3.59E-05	3.60E-05

1(b) *Graph of Solution and Bar Chart of Errors.*

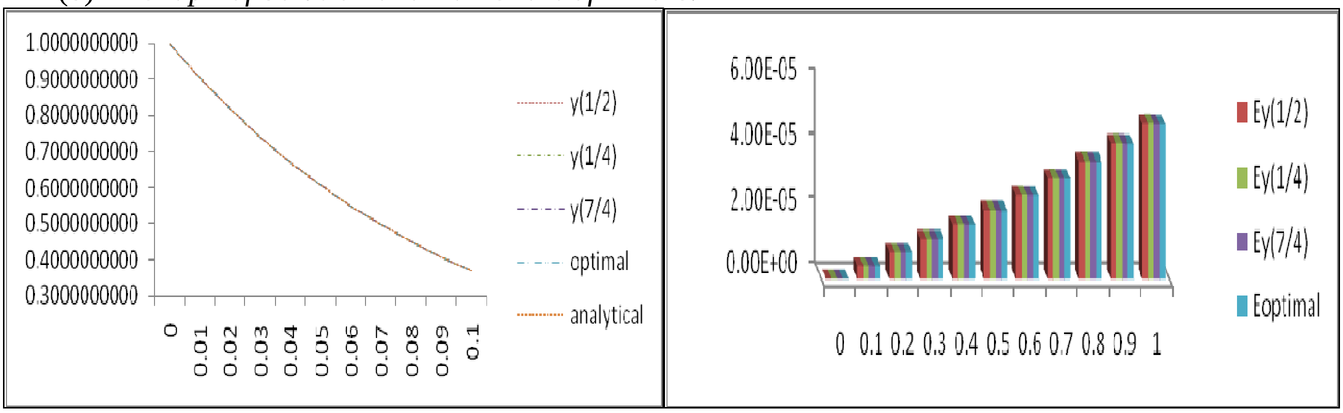


2(a) *Numerical Solution and Tables of Errors*

x	y(1/2)	y(1/4)	y(7/4)	optimal	analytical
0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.01	0.9048333333	0.9048333333	0.9048333333	0.9048333333	0.9048374180
0.02	0.8187225417	0.8187225394	0.8187225439	0.8187225466	0.8187307531
0.03	0.7408057996	0.7408057927	0.7408058057	0.7408058136	0.7408182207
0.04	0.6703032900	0.6703032766	0.6703033020	0.6703033174	0.6703200460
0.05	0.6065094003	0.6065093785	0.6065094197	0.6065094448	0.6065306597
0.06	0.5487856598	0.5487856278	0.5487856883	0.5487857252	0.5488116361
0.07	0.4965543500	0.4965543060	0.4965543892	0.4965544398	0.4965853038
0.08	0.4492927225	0.4492926647	0.4492927739	0.4492928402	0.4493289641
0.09	0.4065277670	0.4065276938	0.4065278322	0.4065279163	0.4065696597
0.1	0.3678314776	0.3678313871	0.3678315581	0.3678316621	0.3678794412

x	Ey(1/2)	Ey(1/4)	Ey(7/4)	Eoptimal
0	0.00E+00	0.00E+00	0.00E+00	0.00E+00
0.01	4.08E-06	4.08E-06	4.08E-06	4.08E-06
0.02	8.21E-06	8.21E-06	8.21E-06	8.21E-06
0.03	1.24E-05	1.24E-05	1.24E-05	1.24E-05
0.04	1.68E-05	1.68E-05	1.67E-05	1.67E-05
0.05	2.13E-05	2.13E-05	2.12E-05	2.12E-05
0.06	2.60E-05	2.60E-05	2.59E-05	2.59E-05
0.07	3.10E-05	3.10E-05	3.09E-05	3.09E-05
0.08	3.62E-05	3.63E-05	3.62E-05	3.61E-05
0.09	4.19E-05	4.20E-05	4.18E-05	4.17E-05
1	4.80E-05	4.81E-05	4.79E-05	4.78E-05

2(b) *Graph of Solution and Bar Chart of Errors.*

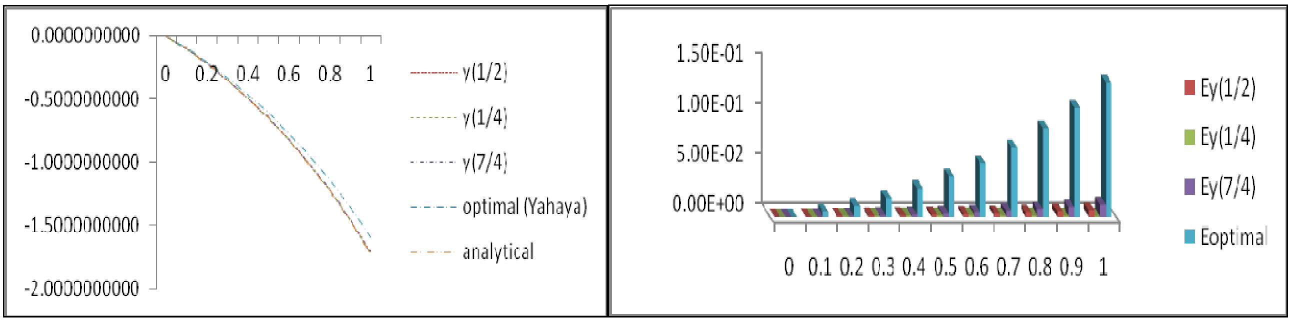


3(a) *Numerical Solution and Tables of Errors*

x	y(1/2)	y(1/4)	y(7/4)	Optimal (Yahaya 2009)	analytical
0	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.1	-0.1048333333	-0.1048333333	-0.1048333333	-0.1048333333	-0.1051709181
0.2	-0.2206806084	-0.2207517773	-0.2206078733	-0.2097763896	-0.2214027582
0.3	-0.3486977969	-0.3489342722	-0.3484633860	-0.3148402337	-0.3498588076
0.4	-0.4901633083	-0.4906787407	-0.4896604103	-0.4200271807	-0.4918246976
0.5	-0.6464898986	-0.6474200912	-0.6455911064	-0.5253376638	-0.6487212707
0.6	-0.8192387372	-0.8207450288	-0.8177929079	-0.6307718694	-0.8221188004
0.7	-1.0101349960	-1.0124081343	-1.0079636772	-0.7363299488	-1.0137527075
0.8	-1.2210850778	-1.2243496271	-1.2179784459	-0.8420120483	-1.2255409285
0.9	-1.4541956546	-1.4587150062	-1.4499079018	-0.9478183139	-1.4596031112
1	-1.7117947063	-1.7178767690	-1.7060388057	-1.0537488913	-1.7182818285

x	Ey(1/2)	Ey(1/4)	Ey(7/4)	Eoptimal
0	0.00E+00	0.00E+00	0.00E+00	0.00E+00
0.1	3.38E-04	3.38E-04	3.38E-04	3.38E-04
0.2	7.22E-04	6.51E-04	7.95E-04	1.16E-02
0.3	1.16E-03	9.25E-04	1.40E-03	3.50E-02
0.4	1.66E-03	1.15E-03	2.16E-03	7.18E-02
0.5	2.23E-03	1.30E-03	3.13E-03	1.23E-01
0.6	2.88E-03	1.37E-03	4.33E-03	1.91E-01
0.7	3.62E-03	1.34E-03	5.79E-03	2.77E-01
0.8	4.46E-03	1.19E-03	7.56E-03	3.84E-01
0.9	5.41E-03	8.88E-04	9.70E-03	5.12E-01
1	6.49E-03	4.05E-04	1.22E-02	6.65E-01

3(b) Graph of Solution and Bar Chart of Errors.

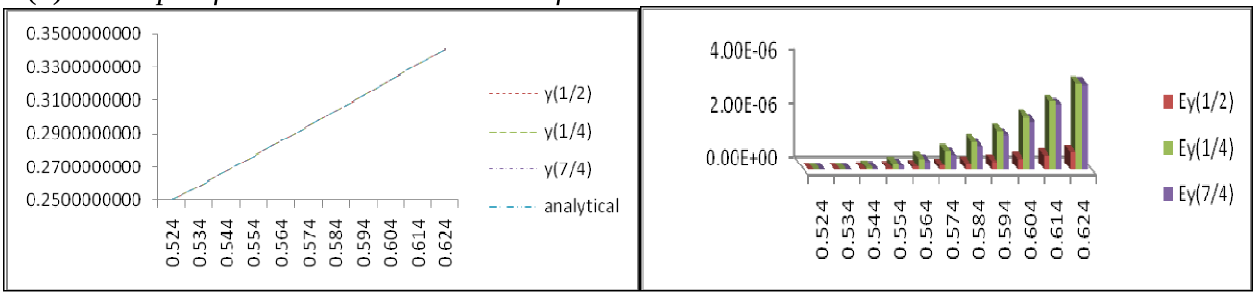


4(a) Numerical Solution and Tables of Errors

x	y(1/2)	y(1/4)	y(7/4)	analytical
0.524	0.2500000000	0.2500000000	0.2500000000	0.2500000000
0.534	0.2587096767	0.2587096767	0.2587096767	0.2587096750
0.544	0.2675158796	0.2675157991	0.2675159099	0.2675158630
0.554	0.2764150870	0.2764148422	0.2764151876	0.2764150415
0.564	0.2854037394	0.2854032452	0.2854039599	0.2854036510
0.574	0.2944782418	0.2944774116	0.2944786414	0.2944780962
0.584	0.3036349648	0.3036337106	0.3036356121	0.3036347474
0.594	0.3128702460	0.3128684789	0.3128712191	0.3128699420
0.604	0.3221803916	0.3221780215	0.3221817783	0.3221799863
0.614	0.3315616780	0.3315586139	0.3315635751	0.3315611561
0.624	0.3410103528	0.3410065030	0.3410128667	0.3410096993

x	Ey(1/2)	Ey(1/4)	Ey(7/4)
0.524	0.00E+00	0.00E+00	0.00E+00
0.534	1.66E-09	1.66E-09	1.66E-09
0.544	1.66E-08	6.39E-08	4.70E-08
0.554	4.55E-08	1.99E-07	1.46E-07
0.564	8.84E-08	4.06E-07	3.09E-07
0.574	1.46E-07	6.85E-07	5.45E-07
0.584	2.17E-07	1.04E-06	8.65E-07
0.594	3.04E-07	1.46E-06	1.28E-06
0.604	4.05E-07	1.96E-06	1.79E-06
0.614	5.22E-07	2.54E-06	2.42E-06
0.624	6.54E-07	3.20E-06	3.17E-06

4(b) Graph of Solution and Bar Chart of Errors.



CONCLUSION

This paper has demonstrated the derivation of a generalized continuous linear multistep method for the direct integration of second order ordinary differential equations without reducing it to systems of first order ordinary differential equation which has been the usual practice. Also, the paper has presented a generalized scheme which generates several block/parallel schemes for substitution of s value. Numerical results show that the schemes generated converges better on some problems than the optimal order method. Continuous Linear Multistep methods are self starting because explicit discrete schemes derived via collocation techniques were used to predict both grid and off-grid points.

Since these methods involve an explicit definition it is however suggested that an additional term f_{n+2} will be added to obtain another class of fully implicit block schemes in a future paper. Also, stability analysis will be carried out which is expected to be dependent on s, this enables the choice of the best method that suites a problem.

REFERENCES

[1] Adee S. O., Onumanyi P., Sirisena U. W. and Yahaya Y. A. (**2005**): *Journal of Computational and Applied Mathematics*, 175, 369 – 373.
[2] Butcher J. C. (**2003**): *Numerical Methods for Ordinary Differential Equations*, John Wiley and sons, New York, 115, 362 – 364.
[3] Ehigie J. O., Okunuga S. A., and Sofoluwe A. B. (**2010**): *Journal of Institute of Mathematics & Computer Sciences*, (Computer Science Ser.) vol. 21, No. 1, 105 - 119.
[4] Ehigie J. O., Okunuga S. A. and Sofoluwe A. B. **(2010)**: *Journal of Institute of Mathematics and Computer Sciences*, Vol 21, No 2, 219 - 230.
[5] Gear C. W. (**1965**): *SIAM J. of Numerical Analysis*, 2, 69 - 86.
[6] Kayode S. J. (**2005**): *An Improved Numerov method for Direct Solution of General Second Order Initial Value Problems of Ordinary Equations*, National Maths Centre proceedings 2005.
[7] Lambert J. D. (**1973**): *Computational Methods in Ordinary Differential Equations*, John Wiley and sons, New York, 253 – 255.
[8] Okunuga S. A. and Onumanyi P. (**1985**): *An Accurate Collocation Method for Solving ODEs'*, AMSE, France 4(4), 45 – 48.
[9] Okunuga S. A. and Onumanyi P. (**1986**): *A Collocation Tau Method for the Solution of ODEs*, Abacus, 17, 51 – 60.
[10] Okunuga S. A. and Ehigie J. O. (**2009**): *Journal of Nigerian Mathematical Physics*, 43 – 50.
[11] Onumanyi P., Awoyemi D. O., Jator S. N. and Sirisena U. W. (**1994**): *J. Nig. Maths Soc.* 13, 37 – 51.
[12] Onumanyi P., Fatokun J., and Adejo B. O. (**2008**): *J. of Math. Soc*. 27, 69 – 90.
[13] Yahaya Y. A. and Badmus A. M. (2009): *African Journal of Mathematics and Computer Science research* vol. 2(4), 069 – 072.