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Burnside method and the great orthogonality theorems on groups of order 8

M. A. Mohammed and Falmata A. Zanna

Department of Mathematics, School of Science Education, Federal College of Education (Technical) Potiskum, Yobe State, Nigeria

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ABSTRACT

Two important method of finding the irreducible representations of group were considered, the Burnside method and the Great Orthogonality Theorem. The irreducible representations of cyclic group of order 8 (C_8) and the dihedral group (D_4) of the same order were obtain using the two methods, and the result were compared. Both method can be used to find the irreducible representations of D_4 and C_8respectively. The Burnside method is lengthy and can be applied to any group, while in the case of the Great Orthogonality Theorem method the groups type need to be identified first.

Key words: Irreducible representation, Character table, Great Orthogonality Theorem, Burnside method, conjugacy classes.

INTRODUCTION

A group is an abstract object; this may create difficulties in dealing with many of its structured problems. Therefore, it is reasonable to seek an automated process for making a bijection between group theory and some other familiar theories. Representation theory is a tool, which reduces group theoretical problems into problems in linear algebra, which is a very well understood theory.

There are five groups of order 8, which consist of two non-abelian and three abelian groups. In this paper, two groups were considered, one abelian that is, the cyclic group of order 8 (C_8) and the other is the dihedral group of the same order (D_4) , which is non-abelian.

MATERIALS AND METHODS

2. PRELIMINARIES

To begin, we shall need some preliminary fact and brief discussion of notation.

2.1 Definition

A representation of a group G with representation space V is a homomorphism

$\rho: g \to \rho(g)$of G into $GL(V)$.

From the homomorphism property we have for $g, h \in G$:

$v\rho(gh) = v\rho(g)\rho(h),$

$v\rho(1) = v1_V$

2.2 Definition
Let $\rho: G \to GL(n, F)$be a representation of a group G over a field F. The function $\chi: G \to$ F defined by $\chi(g) = tr(\rho(g))$is called character of ρ.

The character satisfies the following properties:
1. $\chi_\rho(e) = \deg(\rho)$.
2. $\chi_\rho(xgx^{-1}) = \chi_\rho(g) \forall x, g \in G$.
3. $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$

2.3Burnside Method
Burnside method proposed by Burnside in 1911, are use to obtain the irreducible representations of a group. Using this method, three formulas are involved in finding the irreducible representations of a group that is irreducible representations$k$. Using these formulas, class multiplication coefficients, characters of irreducible representations in term of $d_k$and the numerical values for $d_k$are obtained.

The first step in getting the irreducible representations is to obtain the class multiplication coefficients. Following Cracknell(1968), the result of multiplying together two classes$C_i$and $C_j$is the sum of several classes C_s:

$$C_i C_j = \sum_s c_{ij,s} C_s \tag{1}$$

Where $c_{ij,s}$are the class multiplication coefficients. Using equations (1), the class multiplication coefficients $c_{ij,s}$can be evaluated.

The next step is to obtain the characters of the irreducible representations in term of the kthirreducible representation. Following Burns(1977) the characters are given in the form of:

$$h_i h_j \chi_i^k \chi_j^k = d_k \sum_{s=1}^{r} c_{ij,s} h_s \chi_s^k \tag{2}$$

Where $h_i$the order of the class isC_i, $\chi_i^k$is the character of the elements in class $C_i$in the irreducible representation k, $d_k$is the dimension of the kthirreducible representation,$c_{ij,s}$is the class of multiplication coefficient and $r$is the number of classes in the group.

The last step of getting the irreducible representations I to obtain the numerical values for $d_k$using Craiknell (1968):

$$\sum_{i=1}^{r} h_i \chi_i^j \chi_i^k = N \delta_{jk} \tag{3}$$

Where $N$is the order of the group, $\delta_{jk}$is the Kronecker Delta symbol, which has the value 1 when $i = j$, but has the value 0 when $i \neq j$, $r$is the number of classes in the group, $\chi_i^j$and $\chi_i^k$are the characters of element in class C_i.

2.4 Great Orthogonality Theorem Method
The Great Orthogonality Theorem is given by:

$$\sum_R [\Gamma_i(R)_{mn}] [\Gamma_j(R)_{m'n'}]^* = \frac{h}{\sqrt{l_i l_j}} \delta_{ij} \delta_{mm'} \delta_{nn'} \tag{4}$$

Where h is the order of the group, $l_i$is the dimension of the ith representation, which is the order of each of the matrices which constitute it, R is the generic symbol given to the various operations in the group, $\Gamma_i(R)_{mn}$is the element in the mth row and the nth column of the matrix corresponding to an operation R in the ith irreducible representation.The three simpler equations obtained from Great Orthogonality Theorem are as follows:

$$\sum_R \Gamma_i(R)_{mn} \Gamma_j(R)_{mn} = 0 \text{ if } i \neq j \tag{5}$$

$$\sum_R \Gamma_i(R)_{mn} \Gamma_i(R)_{m'n'} = 0 \quad \text{if } m \neq m' \text{ and/or } n \neq n' \tag{6}$$

$$\sum_R \Gamma_i(R)_{mn} \Gamma_j(R)_{mn} = h/l_i \tag{7}$$

There are five important rules to find irreducible representations and their characters.

1. The sum of the squares of the dimensions of the irreducible representations of a group is equal to the order of the group, that is,

$$\sum l_i^2 = l_1^2 + l_2^2 + \cdots = h \tag{8}$$

Where l_i is the dimension of the ith representation and h is the order of a group

2. The sum of the squares of the characters in any irreducible representation equal to the order of the group, that is

$$\sum_R [\chi_i(R)]^2 = h \tag{9}$$

Proof: From (4) we may write

$$\sum_R \Gamma_i(R)_{mn}\Gamma_i(R)_{m'm'} = \frac{h}{l_i}\delta_{mm'}$$

Summing the left hand side over m and m', we obtain

$$\sum_{m'}\sum_m\sum_R \Gamma_i(R)_{mn}\Gamma_i(R)_{m'm'} = \frac{h}{l_i}\delta_{mm'}$$

$$= \sum_R[\sum_R \Gamma_i(R)_{mn} \sum_R \Gamma_i(R)_{m'n'}]$$

$$= \sum_R \chi_i(R)\chi_i(R)$$

$$= \sum_R [\chi_i(R)]^2$$

While summing the right side over m and m', we obtain

$$\frac{h}{l_i}\sum_{m'}\sum_m \delta_{mm'} = \frac{h}{l_i} l_i = h$$

3. The vectors whose components are characters of two different irreducible representations are orthogonal, that is,

$$\sum_R \chi_i(R)\chi_i(R) = 0 \text{ when } i \neq j$$

Proof: Setting $m = n$ in (5), we obtain

$$\sum_R \chi_i(R)_{mm}\chi_i(R)_{mm} = 0 \text{ if } i \neq j$$

$$\sum_R \chi_i(R)\chi_i(R) = \sum_R[\sum_m \Gamma_i(R)_{mm}\Gamma_j(R)_{mm}]$$

$$= \sum_m[\sum_R \Gamma_i(R)_{mm}\Gamma_j(R)_{mm}] = 0$$

4. In a given representation (reducible or irreducible), the characters of all matrices belonging to operations in the same class are identical.

5. The number of irreducible representations of a group is equal to the number of classes in the group.

There is specific method to find the irreducible representations for cyclic groups using this method. A cyclic group is abelian and each of its h elements is in a separate class. It also has h 1-dimensional irreducible representations. In order to obtain the irreducible representations for a cyclic group, the exponential below is used as the pth irreducible representation, $\Gamma_p(C_n)$:

$$\epsilon^p = \exp\left(\frac{2\pi ip}{8}\right) = \cos\left(\frac{2\pi ip}{8}\right) + i\sin\left(\frac{2\pi ip}{8}\right) \tag{10}$$

RESULTS

Illustration 3.1(Burnside Method)
Irreducible representations of Dihedral group, D_4

D_4 ={(1), (1 3),(2 4),(12)(34),(13)(24), (1 4)(2 3), (1234),(1432) }

The conjugacy classes of D_4 are:
C_1 = (1), C_2 = (12), (34), C_3 = (12)(34), C_4 =(13)(24) (14)(23) andC_5 = (1234), (1432).

The first step to obtain the irreducible representations is to use equation (1) to obtain the class multiplication coefficients. For instance, the class C_5 has two elements, namely (1234) and (1432). The multiplication table of $C_5$is shown below

Table I: The multiplication table of C_5

	(1234)	(1432)
(1234)	(1,3)(2,4)	(1)
(1432)	(1)	(1,3)(2,4)

Since 1 is the element in class $C_1$and (1,3)(2,4) is the element of the class C_3, the table shows that $C_5.C_5 = 2C_1 + 2C_3$.

Therefore, from (1),

$C_5.C_5 = c_{55.1}C_1 + c_{55.2}C_2 + c_{55.3}C_3 + c_{55.4}C_4 + c_{55.5}C_5$, which gives
$2C_1 + 2C_3 = c_{55.1}C_1 + c_{55.2}C_2 + c_{55.3}C_3 + c_{55.4}C_4 + c_{55.5}C_5$.

This implies $c_{55.1} = 2$ and $c_{55.3} = 2$

Evaluating equation (1) for all cases, the non-zero class multiplicationcoefficients are obtained as follows:

$c_{11.1}$=1	$c_{22.1}$=2	$c_{33.1}$=1	$c_{44.1}$=2	$c_{55.1}$=2
$c_{12.2}$=1	$c_{22.3}$=2	$c_{34.4}$=1	$c_{44.3}$=2	$c_{55.3}$=2
$c_{13.3}$=1	$c_{23.2}$=1	$c_{35.5}$=1	$c_{45.2}$=2	
$c_{14.4}$=1	$c_{24.5}$=2			
$c_{15.5}$=1	$c_{25.4}$=2			

Next, the characters of the irreducible representations in term of $d_k$are found using equation (2). For example, in the case $i = j = 1$:

$$h_1 h_1 \chi_1^k \chi_1^k = d_k \sum_{s=1}^{5} c_{11.s}\, h_s \chi_s^k$$

$$= d_k(c_{11.1}h_1\chi_1^k + c_{11.2}h_2\chi_2^k + c_{11.3}h_3\chi_3^k + c_{11.4}h_4\chi_4^k + c_{11.5}h_5\chi_5^k)$$
$$= d_k(c_{11.1}h_1\chi_1^k + (0)h_2\chi_2^k + (0)h_3\chi_3^k + (0)h_4\chi_4^k + (0)h_5\chi_5^k)$$
$$= d_k c_{11.1} h_1 \chi_1^k.$$

Since $c_{11.1} = 1,\ h_1 = 1$, thus $\chi_1^k = d_k$.

Similarly, calculations in the case $i = j = 3$ will yield$\chi_3^k = \pm d_k$. Considering all calculations in cases when $c_{11.s} \neq 0, i, j, s = 1, ..5$, the following results were obtained. For negative value of χ_3^k, the value of $\chi_2^k$and χ_4^kturn out to be 0. For positive values of χ_3^k, we get the following results:

i. If $\chi_2^k = d_k,\ \chi_4^k = d_k$, then $\chi_5^k = d_k$,
ii. If $\chi_2^k = d_k,\ \chi_4^k = -d_k$, then $\chi_5^k = -d_k$,
iii. If $\chi_2^k = -d_k,\ \chi_4^k = d_k$, then $\chi_5^k = -d_k$,
iv. If $\chi_2^k = d_k, -\chi_4^k = -d_k$, then $\chi_5^k = d_k$,

All of these characters of the irreducible representations of $D_4$are shown in the table below, where entries in row $i(i = 1, \ldots 5)$correspond to the irreducible representation.

Table II: Characters of the irreducible representations of D_4in term of d_k

C_1	C_2	C_3	C_4	C_5
d_k	d_k	d_k	d_k	d_k
d_k	d_k	d_k	-d_k	-d_k
d_k	-d_k	d_k	d_k	-d_k
d_k	-d_k	d_k	-d_k	d_k
d_k	0	-d_k	0	0

Lastly, equation (3) is used to obtained the numerical values for d_k.
For each $1 \leq k \leq 5$.

$$\sum_{i=1}^{5} h_i\,(\chi_i^k)^2 = h_1\chi_1^k\chi_1^k + h_2\chi_2^k\chi_2^k + h_3\chi_3^k\chi_3^k + h_4\chi_4^k\chi_4^k + h_5\chi_5^k\chi_5^k$$
$$=\chi_1^k\chi_1^k + 2\chi_2^k\chi_2^k + \chi_3^k\chi_3^k + 2\chi_4^k\chi_4^k + 2\chi_5^k\chi_5^k = 8.$$

For example, using the character of the third irreducible representation, when $k = 3$.

$$\sum_{i=1}^{5} h_i\,(\chi_i^k)^2 = d_3d_3 + 2(-d_3)(-d_3) + d_3d_3 + 2d_3d_3 + 2(-d_3)(-d_3)$$
$$= 8{d_3}^2 = 8.$$

Thus, $d_3 = 1$

Therefore, $d_k = 1$ for the first four irreducible representations and d_k=2 for the fifth irreducible representation. Thus for D_4there are five irreducible representations.

Table III: Character Table of D_4

	C_1	C_2	C_3	C_4	C_5
Γ_1	1	1	1	1	1
Γ_2	1	1	1	-1	-1
Γ_3	1	-1	1	1	-1
Γ_4	1	-1	1	-1	1
Γ_5	2	0	-2	0	0

Illustration 3.2 (Burnside method)
Irreducible representations of Cyclic group,C_8

The group C_8have 8 conjugacy classes as listed below:
$C_1 = (1),\ C_2 = (1,2,3,4,5,6,7,8), C_3 = (1,3,5,7)(2,4,6,8), C_4 = (1,4,7,2,5,8,3,6),$
$C_5 = (1,5)(2,6)(3,7)(4,8),\ C_6 = (1,6,3,8,5,2,7,4), C_7 = (1,7,5,3)(2,8,6,4), C_8 = (1,8,7,6,5,4,3,2)$

Evaluating equation (1) for all cases, the non-zero class multiplication coefficient are as follows:

$c_{11.1}$ =1 $c_{22.3}$ =1 $c_{33.5}$ =1 $c_{44.7}$ =1 $c_{55.1}$ =1 $c_{66.3}$ =1 $c_{77.5}$ =1 $c_{88.7}$ =1
$c_{12.2}$ =1 $c_{23.4}$ =1 $c_{34.6}$ =1 $c_{45.8}$ =1 $c_{56.2}$ =1 $c_{67.4}$ =1 $c_{78.6}$ =1
$c_{13.3}$ =1 $c_{24.5}$ =1 $c_{35.7}$ =1 $c_{46.1}$ =1 $c_{57.3}$ =1 $c_{68.5}$ =1
$c_{14.4}$ =1 $c_{25.6}$ =1 $c_{36.8}$ =1 $c_{47.2}$ =1 $c_{58.4}$ =1
$c_{15.5}$ =1 $c_{26.7}$ =1 $c_{37.1}$ =1 $c_{48.3}$ =1
$c_{16.6}$ =1 $c_{27.8}$ =1 $c_{38.2}$ =1
$c_{17.7}$ =1 $c_{28.1}$ =1
$c_{18.8}$ =1

Next, the character of the irreducible representations in term of $d_k$are found using equation (2). For example, in the case $i = j = 1$:

$$h_1h_1\chi_1^k\chi_1^k = d_k \sum_{s=1}^{8} c_{11.s}\,h_s\chi_s^k$$
$$= d_k(c_{11.1}h_1\chi_1^k + c_{11.2}h_2\chi_2^k + c_{11.3}h_3\chi_3^k + c_{11.4}h_4\chi_4^k + c_{11.5}h_5\chi_5^k$$
$$+c_{11.6}h_6\chi_6^k + c_{11.7}h_7\chi_7^k + c_{11.8}h_8\chi_8^k)$$

$d_k((1)h_1\chi_1^k + (0)h_2\chi_2^k + (0)h_3\chi_3^k + (0)h_4\chi_4^k + (0)h_5\chi_5^k$
$+(0)h_6\chi_6^k + (0)h_7\chi_7^k + (0)h_8\chi_8^k)$
$= d_k h_1 \chi_1^k.$
Since $h_1 = 1$, thus $\chi_1^k = d_k$

$$h_5 h_5 \chi_5^k \chi_5^k = d_k \sum_{s=1}^{8} c_{55.s}\, h_s \chi_s^k$$

$= d_k(c_{55.1}h_1\chi_1^k + c_{55.2}h_2\chi_2^k + c_{55.3}h_3\chi_3^k + c_{55.4}h_4\chi_4^k + c_{55.5}h_5\chi_5^k$
$+c_{55.6}h_6\chi_6^k + c_{55.7}h_7\chi_7^k + c_{55.8}h_8\chi_8^k)$
$d_k((1)h_1\chi_1^k + (0)h_2\chi_2^k + (0)h_3\chi_3^k + (0)h_4\chi_4^k + (0)h_5\chi_5^k$
$+(0)h_6\chi_6^k + (0)h_7\chi_7^k + (0)h_8\chi_8^k)$
$= d_k h_1 \chi_1^k.$
Since $h_1 = h_5 = 1$ and $\chi_1^k = d_k$, thus
$\chi_5^k \chi_5^k = d_k^2,$
$\chi_5^k = \pm d_k$

Considering all cases when $c_{11.s} \neq 0, i, j, s = 1, \ldots 8,$
If $\chi_5^k = d_k$, then $\chi_3^k = \pm d_k$;
If $\chi_5^k = -d_k$, then $\chi_3^k = \pm d_k i$;
If $\chi_3^k = d_k$, then $\chi_6^k = \pm d_k$;
If $\chi_3^k = -d_k$, then $\chi_6^k = \pm d_k i$;
If $\chi_3^k = d_k i$, then $\chi_6^k = \pm d_k \epsilon$; where $\epsilon = i^{\frac{1}{2}}$;
If $\chi_3^k = -d_k i$, then $\chi_6^k = \pm d_k \epsilon^*$; where $\epsilon^* = (-i)^{\frac{1}{2}}$
And the other can be similarly be shown.

All of these characters of the irreducible representations of $C_8$are shown in the table below, where entries in row $i(i = 1, \ldots 8)$correspond to the ith irreducible representation.Lastly, equation (3) is used to obtained the numerical values for d_k. For each $1 \leq k \leq 8$.

$$\sum_{i=1}^{8} h_i\,(\chi_i^k)^2 = h_1\chi_1^k\chi_1^k + h_2\chi_2^k\chi_2^k + h_3\chi_3^k\chi_3^k + h_4\chi_4^k\chi_4^k + h_5\chi_5^k\chi_5^k$$

$+h_6\chi_6^k\chi_6^k + h_7\chi_7^k\chi_7^k + h_8\chi_8^k\chi_8^k$
$=\chi_1^k\chi_1^k + \chi_2^k\chi_2^k + \chi_3^k\chi_3^k + \chi_4^k\chi_4^k + \chi_5^k\chi_5^k + \chi_6^k\chi_6^k + \chi_7^k\chi_7^k + \chi_8^k\chi_8^k = 8.$

Table IV: Character Table of C_8in term of d_k

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
d_k	d_k	d_k	d_k	d_k	d_k	d_k	d_k
d_k	$-d_k$	d_k	$-d_k$	d_k	$-d_k$	d_k	$-d_k$
d_k	$d_k i$	$-d_k$	$-d_k i$	d_k	$d_k i$	$-d_k$	$-d_k i$
d_k	$-d_k i$	$-d_k$	$d_k i$	d_k	$-d_k i$	$-d_k$	$d_k i$
d_k	$d_k\epsilon$	$d_k i$	$-d_k\epsilon^*$	$-d_k$	$-d_k\epsilon$	$-d_k i$	$d_k\epsilon^*$
d_k	$-d_k\epsilon$	$d_k i$	$d_k\epsilon^*$	$-d_k$	$d_k\epsilon$	$-d_k i$	$-d_k\epsilon^*$
d_k	d_k	$-d_k i$	$-d_k\epsilon$	$-d_k$	$-d_k\epsilon^*$	d_k	$d_k\epsilon$
d_k	$-d_k\epsilon^*$	$-d_k i$	$d_k\epsilon$	$-d_k$	$d_k\epsilon^*$	d_k	$-d_k\epsilon$

From the second to sixth irreducible representations, it is necessary to take the complex conjugate of χ_i^jsince complex numbers are involved. For example, using the characters of the fifth irreducible representation, when $k = 5$,

$\sum_{i=1}^{8} h_i\,(\chi_i^k)^2 = (d_5)(\overline{d_5}) + (d_5 i)(\overline{d_5 i}) + (-d_5\epsilon^*)\overline{(-d_5\epsilon^*)} +(-d_5)(\overline{-d_5})$
$+(-d_5\epsilon)(-d_5\epsilon) + (-d_5 i)(\overline{-d_5 i})+(-d_5\epsilon^*)\overline{(-d_5\epsilon^*)}$
$=d_5^2(1 + (\epsilon)(\bar{\epsilon}) + (i)(\bar{i}) + (-\epsilon^*)(\overline{-\epsilon^*}) + (-1)(\overline{-1}) + (-\epsilon)(\overline{-\epsilon})$
$+(-i)(\overline{-i}) + (\epsilon^*)(\overline{\epsilon^*})=8.$
Since $(\epsilon)(\bar{\epsilon}) = 1$ $(-\epsilon^*)(\overline{-\epsilon^*}) = 1$ and $(i)(\bar{i}) = 1,$
$8d_5^2 = 8,$
$d_5 = 1.$
For this group, $d_k = 1,\ 1 \leq k \leq 8$, for all eight irreducible representations. Thus, the character Table of the irreducible representations of $C_8$is given below:

Table V: Character Table C_8

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
Γ_1	1	1	1	1	1	1	1	1
Γ_2	1	-1	1	-1	1	-1	1	-1
Γ_3	1	i	-1	$-i$	1	i	-1	$-i$
Γ_4	1	$-i$	-1	i	1	$-i$	-1	i
Γ_5	1	ϵ	i	$-\epsilon^*$	-1	$-\epsilon$	$-i$	ϵ^*
Γ_6	1	$-\epsilon$	i	ϵ^*	-1	ϵ	$-i$	$-\epsilon^*$
Γ_7	1	ϵ^*	$-i$	$-\epsilon$	-1	$-\epsilon^*$	i	ϵ
Γ_8	1	$-\epsilon^*$	$-i$	ϵ	-1	ϵ^*	i	$-\epsilon$

Illustration 3.3 (Great Orthogonality method)
Irreducible representations of Dihedral group, D_4
D_4 ={(1), (1 3),(2 4),(12)(34),(13)(24), (1 4)(2 3), (1234),(1432) }

The conjugacy classes of D_4 are:
C_1 = (1), C_2 = (12), (34), C_3 = (12)(34), C_4 =(13)(24) (14)(23) and C_5 = (1234), (1432).

According to rule 5, there are five irreducible representations for the group D_4. By rule 1, we find a set of five positive integers, $l_1, l_2, l_3, l_4$and l_5 which satisfy the equation ${l_1}^2 + {l_2}^2 + {l_3}^2 + {l_4}^2 + {l_5}^2 = 8$. The only values of $l_i (i = 1, \dots 5)$ which satisfy this requirement are 1, 1, 1, 1 and 2. Thus, the group D_4has four 1-dimensional irreducible representation and one 2-dimensional irreducible representation. By rule 2, in any group, there will be a 1-dimensional irreducible representations whose characters are all equal to 1, since

$$\sum_R (\chi_1(R))^2 = (1)1^2 + (2)1^2 + (1)1^2 + (2)1^2 + (2)1^2 = 8.$$

The other representations will have to be such that $\sum_R (\chi_1(R))^2 = 8$, which can be true if and only if each $\chi_i(R) = \pm 1$. By rule 3, each of the other three representations has to be orthogonal to the first irreducible representation, Γ_1. Thus, there will have to be two +1's and two -1's. The fifth representation will be of dimension 2, hence $\chi_5(C_1) = 2$. In order to find out the values of $\chi_5(C_2), \chi_5(C_3), \chi_5(C_4)$ and $\chi_5(C_5)$, the orthogonality relationships in rule (3):

$$\sum_R (\chi_1(R))^2 = (1)(2) + (1)\chi_5(C_2) + (1)\chi_5(C_3) + (1)\chi_5(C_4) + (1)\chi_5(C_5) = 0$$
$$\sum_R (\chi_2(R))^2 = (1)(2) + (1)\chi_5(C_2) + (1)\chi_5(C_3) + (-1)\chi_5(C_4) + (-1)\chi_5(C_5) = 0$$
$$\sum_R (\chi_3(R))^2 = (1)(2) + (-1)\chi_5(C_2) + (1)\chi_5(C_3) + (1)\chi_5(C_4) + (-1)\chi_5(C_5) = 0$$
$$\sum_R (\chi_4(R))^2 = (1)(2) + (-1)\chi_5(C_2) + (1)\chi_5(C_3) + (-1)\chi_5(C_4) + (1)\chi_5(C_5) = 0$$

Gives $\chi_5(C_2) = \chi_5(C_4) = \chi_5(C_5) = 0$ and $\chi_5(C_3) = -2$

The complete set of irreducible representations of $D_4$is found to be:

Table VI: Character Table of D_4

	C_1	C_2	C_3	C_4	C_5
Γ_1	1	1	1	1	1
Γ_2	1	1	1	-1	-1
Γ_3	1	-1	1	1	-1
Γ_4	1	-1	1	-1	1
Γ_5	2	0	-2	0	0

Illustration 3.4 (Great Orthogonality method)
Irreducible representations of Cyclic group,C_8

The group C_8have 8 conjugacy classes as listed below:
$C_1 = (1),\ C_2 = (1,2,3,4,5,6,7,8), C_3 = (1,3,5,7)(2,4,6,8), C_4 = (1,4,7,2,5,8,3,6),$
$C_5 = (1,5)(2,6)(3,7)(4,8),\ C_6 = (1,6,3,8,5,2,7,4), C_7 = (1,7,5,3)(2,8,6,4), C_8 = (1,8,7,6,5,4,3,2)$

According to rule 5, there are eight irreducible representations for the group C_8. By rule 1, we find a set of eight positive integers, $l_1, l_2, l_3, l_4, l_5, l_6, l_7$and l_5 which satisfy the equation$l_1{}^2 + l_2{}^2 + l_3{}^2 + l_4{}^2 + l_5{}^2 + l_6{}^2 + l_7{}^2 + l_8{}^2 = 8$. The only values of

$l_i(i = 1, \ldots 8)$ which satisfy this requirement are all 1s. Thus, the group C_8has eight

1-dimensional irreducible.

Since $C_8$is a cyclic group, the exponential below is used:

$\Gamma_{\text{p}}(C_8) = \epsilon^p$= exp $(\frac{2\pi ip}{8})$
= cos $\left(\frac{2\pi ip}{8}\right) + isin(\frac{2\pi ip}{8})$
Abbreviating these exponentials, as ϵ^p[ie, exp $(\frac{2\pi i}{8})$], we write the first column of the following table:

Table VII: Multiplication Table of C_8in terms of ϵ.

	C$_1$	C$_2$	C$_3$	C$_4$	C$_5$	C$_6$	C$_7$	C$_8$
Γ_1	ϵ^8	ϵ	ϵ^2	ϵ^3	ϵ^4	ϵ^5	ϵ^6	ϵ^7
Γ_2	ϵ^{16}	ϵ^2	ϵ^4	ϵ^6	ϵ^8	ϵ^{10}	ϵ^{12}	ϵ^{14}
Γ_3	ϵ^{24}	ϵ^3	ϵ^6	ϵ^9	ϵ^{12}	ϵ^{15}	ϵ^{18}	ϵ^{21}
Γ_4	ϵ^{32}	ϵ^4	ϵ^8	ϵ^{12}	ϵ^{16}	ϵ^{20}	ϵ^{24}	ϵ^{28}
Γ_5	ϵ^{40}	ϵ^5	ϵ^{10}	ϵ^{15}	ϵ^{20}	ϵ^{25}	ϵ^{30}	ϵ^{35}
Γ_6	ϵ^{48}	ϵ^6	ϵ^{12}	ϵ^{18}	ϵ^{24}	ϵ^{30}	ϵ^{36}	ϵ^{42}
Γ_7	ϵ^{56}	ϵ^7	ϵ^{14}	ϵ^{21}	ϵ^{28}	ϵ^{35}	ϵ^{42}	ϵ^{49}
Γ_8	ϵ^{64}	ϵ^8	ϵ^{16}	ϵ^{24}	ϵ^{32}	ϵ^{40}	ϵ^{48}	ϵ^{56}

The remaining columns follows from the group multiplication. It will now be shown that these representation satisfy the orthonormalization condition of equation (5).Consider any two representations, say $\Gamma_p$and $\Gamma_q$where $p - q = r$. The left-hand side of equation (5) take the form

$$(\epsilon^p)^*\epsilon^{p+r}+(\epsilon^{2p})^*\epsilon^{2(p+r)}+(\epsilon^{3p})^*\epsilon^{3(p+r)}+(\epsilon^{4p})^*\epsilon^{4(p+r)}+(\epsilon^{5p})^*\epsilon^{5(p+r)}+(\epsilon^{6p})^*\epsilon^{6(p+r)}$$
$$+(\epsilon^{7p})^*\epsilon^{7(p+r)}+(\epsilon^{8p})^*\epsilon^{8(p+r)}$$

Which can be simplified as

$$\epsilon^r+\epsilon^{2r}+\epsilon^{3r}+\epsilon^{4r}+\epsilon^{5r}+\epsilon^{6r} + \epsilon^{7r}+\epsilon^{8r} = \sum_{s=1}^{8} \exp\left(\frac{2\pi is}{8}\right) \qquad (11)$$

The representations are normalized, since if $\Gamma_p = \Gamma_q$, then $r = 0$ and equation (11) is eight times ϵ^0, that is 8. If $\Gamma_p$and $\Gamma_q$are different, r is some number from 1 to 7 since $\epsilon^8$is equal to 1.

Therefore, the sum of equation (11) reduces to 0, that is,

$$\sum_{s=1}^{8} \exp\left(\frac{2\pi is}{8}\right) = 0$$

Using trigonometric identities
$\sum_{s=1}^{l} \cos\left(\frac{2\pi s}{l}\right) = 0$and $\sum_{s=1}^{l} \sin\left(\frac{2\pi s}{l}\right) = 0$

Reducing the powers of ϵ's to modulo 8, we obtain the table below:

Table VIII: Reducing The Powers of ϵ's to Modulo 8

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
Γ_1	1	ϵ	ϵ^2	ϵ^3	ϵ^4	ϵ^5	ϵ^6	ϵ^7
Γ_2	1	ϵ^2	ϵ^4	ϵ^6	1	ϵ^2	ϵ^4	ϵ^6
Γ_3	1	ϵ^3	ϵ^6	ϵ^1	ϵ^4	ϵ^7	ϵ^2	ϵ^5
Γ_4	1	ϵ^4	1	ϵ^4	1	ϵ^4	1	ϵ^4
Γ_5	1	ϵ^5	ϵ^2	ϵ^7	ϵ^4	ϵ^1	ϵ^6	ϵ^3
Γ_6	1	ϵ^6	ϵ^4	ϵ^2	1	ϵ^6	ϵ^4	ϵ^2
Γ_7	1	ϵ^7	ϵ^6	ϵ^5	ϵ^4	ϵ^3	ϵ^2	ϵ
Γ_8	1	1	1	1	1	1	1	1

Using equation (10) where ϵ^2 is replaced by $i, \epsilon^3$by $-\epsilon^*, \epsilon^4$, by -1, ϵ^5, by $-\epsilon$, ϵ^6, by $-i$, ϵ^7, by ϵ^*, ϵ^8, by 1 and rearranging the rows, the character table of $C_8$is obtained the irreducible representations of C_8

The complete set of irreducible representations of $C_8$is :

Table IX: Character Table C_8

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
Γ_1	1	1	1	1	1	1	1	1
Γ_2	1	-1	1	-1	1	-1	1	-1
Γ_3	1	i	-1	$-i$	1	i	-1	$-i$
Γ_4	1	$-i$	-1	i	1	$-i$	-1	i
Γ_5	1	ϵ	i	$-\epsilon^*$	-1	$-\epsilon$	$-i$	ϵ^*
Γ_6	1	$-\epsilon$	i	ϵ^*	-1	ϵ	$-i$	$-\epsilon^*$
Γ_7	1	ϵ^*	$-i$	$-\epsilon$	-1	$-\epsilon^*$	i	ϵ
Γ_8	1	$-\epsilon^*$	$-i$	ϵ	-1	ϵ^*	i	$-\epsilon$

4.0 Comparison of the two methods

Both the Burnside method and Great Orthogonality Theorem method can be used to obtain the irreducible representations of $D_4$and C_8as shown in table III and VI, V and VIII respectively. Using Burnside method, three formulas are involved to find the irreducible representations. The first and the second formulas are quit lengthy since there is $n!$ calculations for each formula for a group with n classes. Beside, every equation has to be satisfied in the second formula to find the characters of the irreducible representations in term of d_k.

For the Great Orthogonality Theorem method, it has five important rules concerning irreducible representations and their characters are used. In general, this method is not lengthy as Burnside method. Therefore, in order to to deduce the irreducible representations for groups using Great Orthogonality Theorem, the type of the groups need to be identified first.

CONCLUSION

Two of the methods to obtain the irreducible representations are the Burnside method and the Great Orthogonality Theorem method. The two method were used to obtain the irreducible representations of $D_4$and $C_8$and comparison of the two method are made. Both method produce same table, but the Burnside is quite lengthy, than the Orthogonality Theorem Method. Burnside method can be applied to any type of groups without having to consider the structure of the group. Great Orthogonality Theorem is not lengthy as Burnside method but there is specific method for cyclic groups in addition to the Great Orthogonality Theorem formula and the five important rules.

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