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Einstein's equations of motion for test particles exterior to spherical distributions of mass whose tensor field varies with time, radial distance and polar angle

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ABSTRACT

In this article the metric tensor exterior to hypothetical spherical distributions of mass whose tensor field varies with time, radial distance and Polar angle is extended to derive equations of motion for test particles in the gravitational field. The time equation is used to derive the expression for the variation of the time on a clock moving in this gravitational field. For pure polar motion, test particles move with velocity that has an inverse dependence on the radial distance. The results show that the introduction of θ *in this field does not alter the inverse dependence of velocity on the radial distance.*

Keywords: Tensor Field, Test Particles, Gravitational Field, Time, Radial Distance, Polar Angle.

INTRODUCTION

In the year 1915 Einstein published his gravitational field equation given by $G_{\mu\nu} = \frac{-8\pi G}{c^4} T_{\mu\nu}$, where G=gravitational constant, $G_{\mu\nu}$ =Einstein's tensor, $T_{\mu\nu}$ = stress energy tensor, and c is the speed of light [1].

Since then the search has been on for the development of their solutions in the space-time of all distributions of mass in nature and their applications to the motion of test particles of non zero rest masses and photon.

The exact solution to this field equation was first constructed in static and pure radial spherical polar coordinates by Schwarzchild in 1916 by considering spherical bodies such as the sun and the stars.

In Schwarzchild's metric, the tensor field varies with radial distance only. It is the metric tensor exterior to an ideal static spherically symmetric body situated in empty space [12]; [10]; [5]. Schwarz child's metric is the mathematically most simple and astrophysically most satisfactory solution of Einstein's geometrical gravitational field equation in the space exterior to a static homogeneous distribution of mass within a spherical region [13]; [14].

Since the earth is not perfectly spherical [9], its field cannot be a function of only the radial distance as assumed by Schwarzschild.

This paper introduces an astrophysical distribution of mass within the region of spherical geometry, whose tensor field varies with time, radial distance and polar angle only. The equation of motion for particles of non zero rest masses are derived for this gravitational field. This research will help in studying astrophysical spherical distribution of mass whose tensor field varies with time, radial distance and polar angle. An example of such distribution is the homogeneous distribution of mass with spherical region, which is rotating with uniform speed about a fixed diameter [8]; [9]; [10].

Theoretical Analysis

Construction of metric tensor and coefficients of affine connection

Schwarzchilds metric is the solution of Einstein's field equations exterior to a static homogenous spherical body [14]; [3];[4] given by

$$g_{00} = 1 + \frac{2}{c^2} f(r) \tag{1}$$

$$g_{11} = -\left[1 + \frac{2}{c^2} f(r)\right]^{-1} \tag{2}$$

$$g_{22} = -r^2 \tag{3}$$

$$g_{33} = -r^2 \sin^2\theta \tag{4}$$

$$g_{\mu\nu} = 0;\ otherwise \tag{5}$$

where c is the speed of light in vacuum. $f(r)$ is an arbitrary function determined by the distribution; it is a function of the radial coordinate r only since the distribution and hence its exterior gravitational field possess spherical symmetry. From the condition that this metric component should reduce to the field of a point mass located at the origin [14] and contains Newton's equation of motion in the gravitational field of the static homogenous spherical body, it follows that $f(r)$ is the Newtonian gravitational scalar potential in the exterior region of the body, defined in this field as

$$f(r) = \frac{GM}{r} \quad , \qquad r > R_0 \tag{6}$$

where G is the universal gravitational constant, M is the mass of the spherical body and R_0 is the radius of the spherical body.

Let us consider an astrophysical mass distribution within spherical geometry in which the tensor field varies with time, radial distance and polar angle. The covariant metric tensor for this distribution of mass or pressure is given as [12];[2].

$$g_{00} = 1 + \frac{2}{c^2} f(t, r, \theta) \tag{7}$$

$$g_{11} = -\left[1 + \frac{2}{c^2} f(t, r, \theta)\right]^{-1} \tag{8}$$

$$g_{22} = -r^2 \tag{9}$$

$$g_{33} = -r^2 \sin^2\theta \tag{10}$$

$$g_{\mu\nu} = 0; otherwise \tag{11}$$

where $f(t, r, \theta)$ is an arbitrary function, determined by the mass or pressure and possesses all the symmetries of the latter. In approximate gravitational fields, it is equal to Newton's gravitational scalar potential exterior to the spherical mass distribution.

To obtain the corresponding contra variant metric tensor for this gravitational field, we impose the Quotient Theorem [6] of the tensor analysis to obtain the components of the contra variant tensor as

$$g^{00} = \left[1 + \frac{2}{c^2} f(t, r, \theta)\right]^{-1} \tag{12}$$

$$g^{11} = -\left[1 + \frac{2}{c^2} f(t, r, \theta)\right] \tag{13}$$

$$g^{22} = -r^{-2} \tag{14}$$

$$g^{33} = -\left(r^2 \sin^2 \theta\right)^{-1} \tag{15}$$

$$g^{\mu\nu} = 0; otherwise \tag{16}$$

The coefficients of affine connection, defined by the metric tensor of space-time are determined using the tensor equation

$$\Gamma^{\delta}_{\mu\lambda} = \frac{1}{2} g^{\delta\nu} \left\{g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda s}\right\} \tag{17}$$

They are found to be given explicitly in terms of $(ct, r, \theta,)$ as

$$\Gamma^{0}_{00} = \frac{1}{2} g^{00}.g_{00.0} \tag{18}$$

$$\Gamma^{0}_{10} = \Gamma^{0}_{01} = \frac{1}{2} g^{00}.g_{00.1} \tag{19}$$

$$\Gamma^{0}_{11} = \frac{1}{2} g^{00}.g_{11.0} \tag{20}$$

$$\Gamma^{0}_{20} = \Gamma^{0}_{02} = \frac{1}{2} g^{00}.g_{00.2} \tag{21}$$

$$\Gamma^{1}_{00} = \frac{1}{2} g^{11}.g_{00.1} \tag{22}$$

$$\Gamma^{1}_{10} = \Gamma^{1}_{01} = \frac{1}{2} g^{11}.g_{11.0} \tag{23}$$

$$\Gamma^{1}_{11} = \frac{1}{2} g^{11}.g_{11.1} \tag{24}$$

$$\Gamma^{1}_{21} = \Gamma^{1}_{12} = \frac{1}{2} g^{11}.g_{11.2} \tag{25}$$

$$\Gamma^{1}_{22} = \frac{1}{2} g^{11}.g_{22.1} \tag{26}$$

$$\Gamma^{1}_{33} = \frac{1}{2} g^{11}.g_{33.1} \tag{27}$$

$$\Gamma^{2}_{00} = \frac{1}{2} g^{22} - g_{00,2} \tag{28}$$

$$\Gamma^{2}_{11} = \frac{1}{2} g^{22}.g_{11.2} \tag{29}$$

$$\Gamma^{2}_{33} = \frac{1}{2} g^{22}.g_{33.2} \tag{30}$$

$$\Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{2} g^{22}.g_{22.1} \tag{31}$$

$$\Gamma^{3}_{31} = \Gamma^{3}_{13} = \frac{1}{2} g^{33}.g_{33.1} \tag{32}$$

where the comma denotes partial differentiation w.r.t (0,1,2)=(ct, r, θ). Equation (18) to (32) can be written explicitly in terms of (ct, r, θ,) as

$$\Gamma_{00}^{0} = \frac{1}{c^2}\left[1+\frac{2}{c^2}f(t,r,\theta)\right]^{-1}\frac{\partial f}{\partial t}(t,r,\theta) \tag{33}$$

$$\Gamma_{10}^{0} = \Gamma_{01}^{0} = \frac{1}{c^2}\left[1+\frac{2}{c^2}(t,r,\theta)\right]^{-1}\frac{\partial f}{\partial r}(t,r,\theta) \tag{34}$$

$$\Gamma_{11}^{0} = \frac{1}{c^2}\left[1+\frac{2}{c^2}f(t,r,\theta)\right]^{-3}\frac{\partial f}{\partial t}(t,r,\theta) \tag{35}$$

$$\Gamma_{20}^{0} = \Gamma_{02}^{0} = \frac{1}{c^2}\left[1+\frac{2}{c^2}f(t,r,\theta)\right]^{-1}\frac{\partial f}{\partial \theta}(t,r,\theta) \tag{36}$$

$$\Gamma_{00}^{1} = \frac{1}{c^2}\left[1+\frac{2}{c^2}f(t,r,\theta)\right]\frac{\partial f}{\partial \theta}(t,r,\theta) \tag{37}$$

$$\Gamma_{10}^{1} = \Gamma_{01}^{1} = -\frac{1}{c^2}\left[1+\frac{2}{c^2}f(t,r,\theta)\right]^{-1}\frac{\partial f}{\partial \theta}(t,r,\theta) \tag{38}$$

$$\Gamma_{11}^{1} = -\frac{1}{c^2}\left[1+\frac{2}{c^2}f(t,r,\theta)\right]^{-1}\frac{\partial f}{\partial \theta}(t,r,\theta) \tag{39}$$

$$\Gamma_{21}^{1} = \Gamma_{12}^{1} = -\frac{1}{c^2}\left[1+\frac{2}{c^2}f(t,r,\theta)\right]^{-1}\frac{\partial f}{\partial \theta}(t,r,\theta) \tag{40}$$

$$\Gamma_{22}^{1} = -r\left[1+\frac{2}{c^2}f(t,r,\theta)\right] \tag{41}$$

$$\Gamma_{33}^{1} = -r\left[1+\frac{1}{c^2}f(t,r,\theta)\right]\sin^2\theta \tag{42}$$

$$\Gamma_{00}^{2} = \frac{r^{-2}}{c^2}\frac{\partial f}{\partial \theta}(t,r,\theta)\text{-} \tag{43}$$

$$\Gamma_{11}^{2} = \frac{r^{-2}}{c^2}\left[1+\frac{2}{c^2}f(t,r,\theta)\right]^{-2}\frac{\partial f}{\partial \theta}(t,r,\theta) \tag{44}$$

$$\Gamma_{33}^{2} = -\sin\theta\cos\theta \tag{45}$$

$$\Gamma_{12}^{2} = \Gamma_{21}^{2} = r^{-1} \tag{46}$$

$$\Gamma_{31}^{3} = \Gamma_{13}^{3} = r^{-1} \tag{47}$$

$$\Gamma_{\beta r}^{\alpha} = 0; otherwise \tag{48}$$

Thus, this gravitational field has 15 non-zero coefficients of affine connection; unlike the Schwarzchild's field which has 9 affine connections. Thus, we expect the gravitational field in this article to have some peculiarities not found in Schwarzchild's field.

Motion of test particles exterior to spherical bodies whose tensor field depends on time, radial distance and polar angle.

A test mass is one which is so small that the gravitational field produced by it is so negligible that it doesn't have any effect on the space metric. A test mass is a continuous body, which is approximated by its geometrical centre; it has nothing in common with a point mass whose density should obviously be infinite [7]. The general relativistic equations of motion for test particles in a gravitational field are given by:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma_{\nu\lambda}^{\mu}\left(\frac{dx^\nu}{d\tau}\right)\left(\frac{dx^\lambda}{d\tau}\right) = 0 \tag{49}$$

where τ is the proper time. We used equation (49) to construct time, radial and polar equations of motion for particles of non-zero rest mass in the gravitational field under consideration. Setting $\mu = 0$ into equation (49) and substituting equations (33) to (36) gives the time equation of motion as

$$\ddot{t} + \frac{2}{c^2}\left[1+\frac{2}{c^2} f(t,r,\theta)\right]^{-1} \frac{\partial f(t,r,\theta)}{\partial r}.\dot{t}\dot{r} + \frac{2}{c^2}\left[1+\frac{2}{c^2} f(t,r,\theta)\right]^{-1} \frac{\partial f}{\partial \theta}(t,r,\theta)\dot{t}\dot{\theta} + \frac{1}{c}\left[1+\frac{2}{c^2} f(t,r,\theta)\right]^{-1} \frac{\partial f}{\partial t}(t,r,\theta)\dot{t}^2 + \frac{1}{c^3}\left[1+\frac{2}{c^2} f(t,r,\theta)\right]^{-3} \frac{\partial f}{\partial t}(t,r,\theta)\dot{r}^2 = 0 \tag{50}$$

$$\frac{\ddot{t}}{\dot{t}} + \frac{2\partial f(t,r,\theta)\dot{r}}{c^2\left[1+\frac{2f}{c^2}(t,r,\theta)\right]} + \frac{2\partial f(t,r,\theta)\theta}{c^2\left[1+\frac{2f}{c^2}(t,r,\theta)\right]} = 0 \tag{51}$$

$$\frac{\partial}{\partial \tau}\left(\ln \dot{t} + \ln\left[1+\frac{2f(t,r,\theta)}{c^2}\right]\right) = 0 \tag{52}$$

Integrating equation (52) gives

$$\dot{t} = \frac{2}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1} \tag{53}$$

Equation (50) is the expression for the variation on the time on a clock with this gravitational field.

Similarly, setting $\mu = 1,2 and 3$ into equation (49) gives the radial, polar and azimuthal equations of motion as (54), (55) and (56) respectively.

$$\ddot{r} + \left[1+\frac{2}{c^2} f(t,r,\theta)\right]\frac{\partial f(t,r,\theta)}{\partial r}\dot{t}^2 - \frac{2}{c}\left[1+\frac{2}{c^2} f(t,r,\theta)\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial t}\dot{t}\dot{r} - \frac{1}{c^2}\left[1+\frac{2}{c^2} f(t,r,\theta)\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\dot{r}^2 - \frac{2}{c^2}\left[1+\frac{2}{c^2} f(t,r,\theta)\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial t}\dot{r}\dot{\theta} - r\left[1+\frac{2}{c^2} f(t,r,\theta)\right]\dot{\theta}^2 - r\left[1+\frac{2}{c^2} f(t,r,\theta)\right]\sin^2\theta\,\dot{\phi} = 0 \tag{54}$$

$$\ddot{\theta} + \frac{1}{r^2}\frac{\partial f(t,r,\theta)}{\partial \theta}\dot{t}^2 - \frac{1}{c^2 r^2}\left[1+\frac{2}{c^2} f(t,r,\theta)\right]^{-2}\frac{\partial f(t,r,\theta)}{\partial \theta}\dot{r}^2 - \sin\theta\cos\theta\,\dot{\phi}^2 + \frac{2}{r}\dot{r}\dot{\theta} = 0 \tag{55}$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0 \tag{56}$$

For pure radial motion $\dot{\theta} = \dot{\phi} \equiv 0$ and hence the radial equation (55) becomes

Equation (57) is the pure radial equation of motion for particles of nonzero rest masses in this field.

Consider the polar motion of test particles that has radial dependence, in this case, equation (55) reduces to

(58)

or

(59)

Integrating equation (58) gives the instantaneous polar velocity as

$$\dot{\phi} = \frac{A}{r^2} \tag{60}$$

where is the constant of integration.

This motion has an inverse square dependence on the radial distance.

Result in this field is similar to the result obtained in our earlier publication when [11]. We can therefore conclude that the introduction of θ to this field does not change the inverse square dependence on the radial distance.

CONCLUSION

The time, radial, polar and Azimuthal equations of motion for test particles exterior to astrophysical real spherical distribution of mass were found to be equations (51), (55), (56) and (57) respectively.

The solution of the time equation of motion gives the variation of the time on a clock with the gravitational field. Thus, the expression for gravitational time dilation in this gravitational field has been obtained as equation (51).

The radial equation of motion given by equation (55) can be used to obtain the instantaneous speed of a particle of nonzero rest mass in this field.

The coefficients of affine connection obtained can be used to construct the Riemann-Christoffel, Ricci and Einstein's tensor for this field and hence the Einstein's field equations for this gravitational field can be obtained, the Einstein's equation contain only a single unknown $f(t,r,\theta)$ and thus can be solved completely to obtain explicit values for $f(t,r,\theta)$. This is opened up for further research.

REFERENCES

[1] J.L. Anderson,Principles of Relativity Physics, Academic Press, New York, **1967**,Pp25-30.
[2] G. Arfken, Mathematical Methods for Physicists, 5th ed., Academic Press, New York,**1995**, Pp467-469.
[3] P.G. Bergmann, Introduction to the Theory of Relativity, Prentice Hall, India,**1987**,Pp203-212.
[4] E.N. Chifu; S.X.K. Howusu, *Journal of NAMP*,**2008**, 13, 363-366.
[5] E.N. Chifu; S.X.K. Howusu;L.W. Lumbi, *Progress in Physics*,**2009**, 3, 18 -23.
[6] R. Dmitri; B. Larissa, *Progress in Physics*,**2008**, 2, 166-168.
[7] P. Dunsby, An Introduction to Tensor and Relativity, Shiva, Cape Town,**2000**, 51-110.
[8] A. Einstein, The Foundation of the General Relativity. *Annalen der Physik*, **1916**.
[9] S.X.K.Howusu, Discourse on general relativity, Jos University Press Ltd., **2003**, 98-105.
[10] S.X.K. Howusu, *Natural Philosophy Society*, Jos,**2007**, 2-227.
[11]W.L. Lumbi;M.S. Liman;R.N. Jatau;I. Umar, *Integrated Journal of Scienceand Engineering*, **2010**, 9(1), 20-25.
[12] K. Schwarzschild, Uber das Gravitationsfeld eines Massen-punktes nach der Einsteinschen Theorie. *Sitzungsberichte der Koniglich Preussischen Akademie der Wissenschaften*, **1916**, 189-196 (published in English as: Schwarzschild K. *Abraham Zelmanov Journal*, **2008**, 1, 10-19).
[13] S. Weinberg, Gravitation and Cosmology, J. Willey and sons. New York, **1972**, 122-127.
[14] Wikipedia, The Free Encyclopaedia, General relativity,**2008**, http://www.en.ikipedia.org