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Generalization of two dimensional canonical SS-transform

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ABSTRACT

The canonical transform (CT) is a widely used integral transform with many applications in applied mathematics, mathematical physics and engineering sciences. The canonical transform (CT) is related to the Fourier transform (FT). The Fourier transform can be generated into the fractional Fourier transform, linear canonical transform. This paper is devoted for analytic study of two dimensional generalized canonical SS-transform, and parsevals identity for two dimensional canonical SS-transform.

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INTRODUCTION

The Fourier transform (FT) is undoubtedly one of the most valuable and frequently used tools in signal processing and analysis [2]. The fractional Fourier transform has been found to have several applications in the areas of optics and filtering etc [4], [5]. Fractional Fourier transforms special case of linear canonical transform, numbers of integral transform are extended in its fractional domain for example, Almeida [1] had studied fractional Fourier transform, Chavhan.S.B and Borkar.V.C [3] developed two dimensional canonical cosine-cosine transform. The linear canonical transform is defined as [6]

$$\{CTf(t)\}(s)=\sqrt{\frac{1}{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty} e^{-i\left(\frac{s}{b}\right)t}\cdot e^{\frac{i}{2}(a/b)t^2} f(t)\,dt,$$

This paper emphasizes define two dimensional canonical SS-transform, and deriving its analyticity theorem then some properties of two dimensional of SS-transform are discussed and finally conclusion is given. Notation and terminology as per zemanian [7],[8].

2 Definition two dimensional (2D) canonical SS- transform:

$$\{2DCSST\ f(t,x)\}(s,w)=\left\langle f(t,x),K_{s_1}(t,x)K_{s_2}(x,w)\right\rangle$$

$$\left\{2DCST\ f(t,x)\right\}(s,w)$$
$$=(-1)\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f(t,x)\,dxdt$$

Where, $K_{s_1}(t,s)=(-i)\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\sin\left(\frac{s}{b}t\right).e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}$ when $b\neq 0$

$=\sqrt{d}\,e^{\frac{i}{2}c\left(ds^2\right)}\delta(t-ds)$ when b = 0

$K_{s_2}(x,w)=(-i)\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\sin\left(\frac{w}{b}x\right).e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}$ when $b\neq 0$

$=\sqrt{d}\,e^{\frac{i}{2}c\left(dw^2\right)}\delta(x-dw)$ when b = 0

where $\gamma_{E,k}\left\{K_{s_1}(t,s)K_{s_2}(x,w)\right\}=\sup_{\substack{-\infty<t<\infty\\ -\infty<x<\infty}}\left|D_t^kD_x^lK_{s_1}(t,s)K_{s_2}(x,w)\right|<\infty$

3 Analyticity Theorem of 2D Canonical SS-Transform

Theorem 3.1 :(Analyticity) Let $f\in E^1(R^n)$ and its two canonical SS- transform be defined by,

$$\{2DCSST\ f(t,x)\}(s,w)$$
$$=-\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}\sin\left(\frac{s}{b}\right)t\sin\left(\frac{w}{b}\right)x\ f(t,x)\,dxdt$$

Then $\{2DCSST\ f(t,x)\}(s,w)$ is analytic on C^n,if the a, b, $\sup pf\subset s_a$ and s_b where $s_a=\left\{t:t\in R^n,\ |t|\leq a,a>0\right\}$, $s_b=\left\{x:x\in R^n,\ |x|\leq b,b>0\right\}$ moreover $\left\{2DCSST\ f(t,x)\right\}(s,w)$ is differentiable and $D_s^kD_w^l\{2DCSST\ f(t,x)\}(s,w)=\left\langle f(t,x),D_s^kD_w^lK_{s_1}(t,s)K_{s_2}(x,w)\right\rangle$

Proof: Let, $s:\{s_1,s_2,\ldots\ldots.s_j.\ldots.s_n\}\in C^n$ and

$w:\{w_1,w_2,\ldots\ldots.w_j.\ldots.w_n\}\in C^n$

We first prove that, $\frac{\partial}{\partial s_j}\frac{\partial}{\partial w_j}\{2DCSST\ f(t,x)\}(s,w)$ exists,

$$\frac{\partial^n}{\partial s_j^n}\frac{\partial^n}{\partial w_j^n}\{2DCSST\ f(t,x)\}(s,w)=\left\langle f(t,x),\frac{\partial^n}{\partial s_j^n}\frac{\partial^n}{\partial w_j^n}K_{s_1}(t,s)K_{s_2}(x,w)\right\rangle \qquad (3.1)$$

we prove the result $n = 1$, the general result following by induction.

For fixed $s_j\neq 0$ choose two concentric circles C and C^1 with centre s_j and radii r and r_1 respectively such that $0<r<r_1<|s_j|$.

Let Δs_j be a complex increment satisfying $0<\left|\Delta s_j\right|<r$. Also for fixed $w_j\neq 0$.*Again* choose two concentric circles C and C_1 with centre w_j and radii r' and r_1' respectively such that $0<r'<r_1'<|w_j|$.

Let Δw_j be a complex increment satisfying $0<\left|\Delta w_j\right|<r'$

Consider,

$$\frac{(2DCSST)(s_j+\Delta s_j,w_j)-(2DCSST)(s_j,w_j)}{\Delta s_j}\frac{(2DCSST)(s_j,w_j+\Delta w_j)-(2DCSST)(s_j,w_j)}{\Delta w_j}$$

$$-\left\langle f(t,x),\frac{\partial}{\partial s_j}\frac{\partial}{\partial w_j}K_{s_1}(t,s)K_{s_2}(x,w)\right\rangle$$

$$=\left\langle f(t,x),\Psi\Delta s_j(t)\Psi\Delta w_j(x)\right\rangle \qquad (3.2)$$

where $\Psi\Delta s_j(t)\Delta w_j(x)=\frac{1}{\Delta s_j}\left[K_{s_1}(t,s_1,s_2\cdots\cdots s_j+\Delta s_j\cdots s_n)-K_{s_1}(t,s)\right]$

$$\frac{1}{\Delta w_j}\left[K_{s_2}(x,w_1,w_2\cdots\cdots w_j+\Delta w_j\cdots w_n)-K_{s_2}(x,w)\right]$$

$$-\frac{\partial^n}{\partial s_j^n}\frac{\partial^n}{\partial w_j^n}K_{s_1}(t,s)K_{s_2}(x,w)$$

For any fixed $(t,x)\in R^n$ and any fixed integer.

$k=(k_1,k_2\cdots\cdots k_n)\in N_0$ and $l=(l_1,l_2\cdots\cdots l_n)\in N_0$

$D_t^kD_x^lK_{s_1}(t,s)K_{s_2}(x,w)$ is analytic inside and on C' and C'_1.

By Cauchy integral formula.

$D_t^kD_x^l\ \Psi\Delta s_j\Delta w_j(t,x)$

$$=\frac{1}{4\pi^2 i^2}D_t^kD_x^l\int_{c'}\int_{c_1'}K_{s_1}(t,s)K_{s_2}(x,w)\left(\frac{1}{\Delta s_j}\left(\frac{1}{z-s_j-\Delta s_j}-\frac{1}{z-s_j}\right)-\frac{1}{(z-s_j)^2}\right)\left(\frac{1}{\Delta w_j}\left(\frac{1}{y-w_j-\Delta w_j}-\frac{1}{y-w_j}\right)-\frac{1}{(y-w_j)^2}\right)dzdy$$

$\overline{s}=(s_1\ldots\ldots\ldots s_{j-1},z,s_{j+1}\ldots\ldots\ldots s_n)$.

$\overline{w}=(w_1\ldots\ldots\ldots w_{j-1},y,w_{j+1}\ldots\ldots\ldots w_n)$.

$$=\frac{\Delta s_j\Delta w_j}{-4\pi^2}\int_{c'}\int_{c_1'}\frac{D_t^kD_x^lK_{s_1}(t,\overline{s})K_{s_2}(x,\overline{w})}{(z-s_j-\Delta s_j)(z-s_j)^2(y-w_j-\Delta w_j)(y-w_j)^2}dzdy,$$

But for all $z\in C'$ and $y\in C_1'$ and (t,x) restricted to a compact subset of R^n,

$D_t^k\ D_x^lK_{s_1}(t,s)K_{s_2}(x,w)$ is bounded by constant Q.

$$\left|D_t^k\ D_x^l\Psi\Delta s_j\Delta w_j(t,x)\right|\le\frac{|\Delta s_j||\Delta w_j|}{4\pi^2}\int_{c'}\int_{c_1'}\frac{Q}{(r_1-r)(r_1)(r_1'-r)(r_1')}|dz||dy|$$

$$\le\frac{|\Delta s_j||\Delta w_j|}{4\pi^2}\frac{Q}{(r_1-r)(r_1)(r_1'-r)(r_1')}$$

Thus as $|\Delta s_j|\to 0$, and $|\Delta w_j|\to 0$, $D_t^k\ D_x^l\Psi\Delta s_j\Delta w_j(t,x)$ tends to zero uniformly on the compact subset of R^n, therefore it follows that $\Psi\Delta s_j\Delta w_j(t,x)$ converges in $E(R^n)$ to zero. Since $f\in E^1$, we conclude (3.2) tends to zero. Therefore $\{2DCSST\ f(t,x)\}(s,w)$ is differentiable with respective s_j and w_j. But this is true for all i, $j=1,2,\ldots\ldots n$ .Hence

$\{2DCSSTf(t,x)\}(s,w)$ is analytic on C^n

and, $D_s^k D_w^l \{2DCSST\ f(t,x)\}(s,w) = \left\langle f(t,x), D_s^k D_w^l\ K_{s_1}(t,s) K_{s_2}(x,w) \right\rangle$.

4 Parsevals Identities for Two dimensional Canonical SS-Transform.

If $f(t,x)$ and $g(t,x)$ are inversion of two dimensional Canonical SS-Transform $\{2DCSST\}(s,w)$ and $\{2DCSST\}(s,w)$ respectively, then

4.1 $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x)\overline{g(t,x)}dxdt = 4\pi^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\{2DCSST\}(s,w)}\{2DCSST\}(s,w)\,dsdw$$

4.2 $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(t,x)|^2\, dxdt = 4\pi^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\{2DCSST\}(s,w)|^2\, dsdw$$

Proof: By definition two dimensional Canonical SS-Transform,

$$\{2DCST\, g(t,x)\}(s,w)$$
$$= (-1)\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} g(t,x)\,dxdt \ldots.(4.1)$$

Using inversion formulae of two dimensional Canonical SS-Transform, $g(t,x)$ is

$$g(t,x)$$
$$= -\sqrt{\frac{2\pi i}{b}}\sqrt{\frac{2\pi i}{b}}\, e^{\frac{-i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{-i}{2}\left(\frac{a}{b}\right)x^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right) e^{\frac{-i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{-i}{2}\left(\frac{d}{b}\right)w^2} \{2DCSST\}(s,w)\,dsdw..(4.2)$$

Taking complex conjugates we get

$$\overline{g(t,x)}$$
$$= -\sqrt{\frac{-2\pi i}{b}}\sqrt{\frac{-2\pi i}{b}}\, e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right) e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} \overline{\{2DCSST\}(s,w)}dsdw.$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x)\overline{g(t,x)}dxdt$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x)\,dxdt\left[-\sqrt{\frac{-2\pi i}{b}}\sqrt{\frac{-2\pi i}{b}}\, e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right) e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} \overline{\{2DCSST\}(s,w)}dsdw.\right]$$

By changing the order of the integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x)\overline{g(t,x)}dxdt$$
$$= -\sqrt{\frac{-2\pi i}{b}}\sqrt{\frac{-2\pi i}{b}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\{2DCSST\}(s,w)}dsdw$$
$$\left(\sqrt{2\pi ib}\sqrt{2\pi ib}\,\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f(t,x)\,dxdt\right)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x)\overline{g(t,x)}dxdt = 4\pi^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\{2DCSST\}(s,w)}\{2DCSST\}(s,w)\,dsdw \ldots..4.3$$

Putting $f(t,x) = g(t,x)$ in 4.3 we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(t,x)|^2\, dxdt = 4\pi^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\{2DCSST\}(s,w)|^2\, dsdw$$

5 Properties of kernel

Kernels of 2D canonical SS- transform satisfied following property

(1) $k_{s_1}(t,s)k_{s_2}(x,w)=k_{s_1}(s,t)k_{s_2}(w,x)$ if a = d

(2) $k_{s_1}(t,s)k_{s_2}(x,w)\neq k_{s_1}(s,t)k_{s_2}(w,x)$ if $a\neq d$

(3) $k_{s_1}(-t,s)k_{s_2}(x,w)=k_{s_1}(t,s)k_{s_2}(-x,w)$

(4) $k_{s_1}(-t,-s)k_{s_2}(x,w)=k_{s_1}(t,s)k_{s_2}(-x,-w)$

Above properties of kernel are simple to prove.

CONCLUSION

In this paper 2D canonical SS- transform is generalized in the distributional sense. The analyticity and parsevals identity are proved, and also properties of kernel are mentioned.

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