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On Improving Euler Methods for Initial Value Problems

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Abstract
Euler introduced the famous Euler method in 1728. As the simplest and the most analyzed numerical integration, it has become the stepping-stone of numerical methods for solving Initial value Problems in Ordinary Differential Equations. There has been considerable efforts to improve on Euler method by increasing its order of accuracy. Recently, in [1], Abraham proposed a new improvement on Euler Method called Modified Improved Modified Euler Method. In this work, we investigate the basic properties of this new method vis-à-vis the older ones. Our analysis show that the method is convergent to order 2 and stable when applied to autonomous Initial Value Problem.

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Key words: Stability, Convergence, Absolute stability, Euler method, Initial Value Problems.

INTRODUCTION

Ordinary Differential Equation often arise from the mathematical modelling of physical phenomena in almost every sphere of human endeavour such as engineering, physical and biological sciences.

Given a function $f\big(x, y\,(x)\big)$ and an "initial value" $y(x_0)$, corresponding to a solution value at x_0, we seek to evaluate numerically the function $y(x)$ satisfying

$$\left.\begin{aligned} y'(x) = f\big(x, y\,(x)\big), x\ \in [x_0, x_{end}],\\ y\,(x_0) = \ y_0 \end{aligned}\right\} \qquad (1)$$

An approximate solution to an Initial Value Problem (IVP) given in (1) above is typically obtained by iterating a set of *difference equations* that approximate the original problem. For this reason, we need to *discretize* the independent variable x.

Let $\{x_i | i = 0, 1, \dots, n\}$

be a mesh over the interval of integration,

$$x_0 < \ x_1 \dots < \ x_{n-1} < \ x_n = T$$

Then $h_i = x_i - x_{i-1}$ is the i^{th} step size. We assume for simplicity a uniform mesh, $h = \frac{T - x_0}{n}$. The recursive application of the difference equation defines a mesh $\{y_i\}$, with each y_i an approximation of the exact solution $y(x_i)$.

The number of instances where an exact solution of (1) can be found by analytical means is very limited. Apparently, only a small class of differential equations possesses analytic solution expressible in terms of known tabulated transcendental functions that satisfy the differential equation as well as the initial conditions. Even when the analytic solutions to certain differential equations are available, their numerical evaluation may be quite difficult. This gave rise to the development of many numerical methods for advancing the solution of IVP (1). In the selection of a good numerical scheme, basic characteristics such as consistency, convergence and stability are paramount. In this article, we study these basic properties in relation to the newly proposed method and the existing ones.

2. Development of Euler Methods

2.1 Famous Euler Method

The historical method of Euler involves computing a discrete set $\{\boldsymbol{y_n}\}$ for arguments $\{\boldsymbol{x_n}\}$ using the difference equation

$$\boldsymbol{EM}: \boldsymbol{y_{n+1} - y_n} \begin{cases} = \boldsymbol{\Phi_E\,(x_n, y_n; h)} \\ = \boldsymbol{hf\,(x_n, y_n)}\,,\ \boldsymbol{n = 1, 2, \ldots, m} \end{cases} \tag{2}$$

where the step size $\boldsymbol{h = x_{n+1} - x_n}$

It is linear in $\boldsymbol{y_n}$ and $\boldsymbol{f_n}$, and being a one-step method, it poses no difficulty when there is need to change from one step size to the other [4, 5, 7, 8]. As the simplest and the most analyzed numerical integration, it has become the impetus for developing numerical algorithms for IVPs in ODEs.

2.2 Modified Euler and Improved Euler Method

There has been a considerable effort to improve on Euler method because of its easy implementation and low computational cost. While studying the relationship between an IVP (1) and, in the case when $\boldsymbol{f}$ is independent of $\boldsymbol{y}$, the integration problem,

$$\boldsymbol{y_1 - y_0 = \int_{x_0}^{x_0 + h} f\,(x) dx}$$

Runge [9], observed that Euler method (2) gives rise to a rather inefficient approximation of the integral by the area of a rectangle of height $\boldsymbol{f\,(x_0)}$. Thus, he says, "it is already much better" to extend the Midpoint rule and the Trapezoidal rule to differential equations by inserting for the missing $\boldsymbol{y} -$ values the results of Euler steps yielding the following methods:

$$\boldsymbol{ME}:\ \boldsymbol{y_{n+1} - y_n} \begin{cases} = \boldsymbol{\Phi_{ME}(x_n, y_n; h)} \\ = \boldsymbol{hf\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hf\,(x_n, y_n)\right)} \end{cases}$$

(3)

$$\boldsymbol{IE}:\ \boldsymbol{y_{n+1} - y_n} \begin{cases} = \boldsymbol{\Phi_{IE}(x_n, y_n; h)} \\ = \boldsymbol{\tfrac{1}{2}h\left(f(x_n, y_n) + f\left(x_n + h, y_n + hf\,(x_n, y_n)\right)\right)} \end{cases}$$

(4)

Method (3) is referred to as the *Modified Euler (ME)* or the *Improved Polygon method, while* (4) is known as the *Improved Euler (IE) method.*

2.3 Improved Modified Euler and Modified Improved Modified Euler Method

In [1], Abraham improved on the Modified Euler by inserting the forward Euler method, in place of $\boldsymbol{y_n}$ in the inner function evaluation of the Modified Euler method. This improvement led to a new method called *Improved Modified Euler (IME) Method*. It is given as,

$$\boldsymbol{IME}: \; \boldsymbol{y_{n+1} - y_n} \begin{cases} = \boldsymbol{\Phi_{IME}(x_n, y_n; h)} \\ = \boldsymbol{hf\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hf\left(x_n + h, y_n + hf(x_n, y_n)\right)\right)} \end{cases}$$

(5)

That is, $\boldsymbol{y_n}$ in ME method (3) was replaced with $\boldsymbol{y_n + hf(x_n, y_n)}$.

However, it was found out that the IME method performed very poorly in comparison with the ME method, with respect to autonomous IVP. Thus, a further improvement was carried out by using $\boldsymbol{y_n + \tfrac{1}{2}hf(x_n, y_n)}$ to replace $\boldsymbol{y_n}$ in IME method (5) to develop,

$$\boldsymbol{MIME}: \; \boldsymbol{y_{n+1} - y_n} \begin{cases} = \boldsymbol{\Phi_{MIME}(x_n, y_n; h)} \\ = \boldsymbol{hf\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hf\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hf(x_n, y_n)\right)\right)} \end{cases}$$

(6)

known as *Modified Improved Modified Euler (MIME) method*

3. Basic Properties of Euler Methods

The properties of the increment function $\boldsymbol{\Phi}$ of the newly proposed *MIME* and older *Euler methods* (2) – (6) are, in general, very crucial to their stability and convergence characteristics. These properties are studied in this section. For any standard IVP of an ODE given by (1), we are interested in finding the solution of $\boldsymbol{y(x)}$ by the *Euler methods* (2) – (6). Whenever the function $\boldsymbol{f}$ does not depend on $\boldsymbol{x}$ the equation (1) is said to be autonomous.

Theorem 3.0.1

: *The existence of such a solution* $\boldsymbol{y(x)}$ *is guaranteed and unique provided that* $\boldsymbol{f(x, y)}$:

- *item is continuous in the infinite strip* $\boldsymbol{\Psi} = \{\boldsymbol{x_0} \leq \boldsymbol{x} \leq \boldsymbol{T},\ |\boldsymbol{y}| < \infty\}$,
- *and is, more specifically, Lipschitz continuous in the dependent variable* $\boldsymbol{y}$ *over the same region* $\boldsymbol{\Psi}$, *i.e.* $\exists$ *a positive constant L such that* $\forall\ (\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}, \hat{\boldsymbol{y}}) \in \boldsymbol{\Psi}$,

$$|\boldsymbol{f(x, y)} - \boldsymbol{f(x, \hat{y})}| \leq \boldsymbol{L}|\boldsymbol{y} - \hat{\boldsymbol{y}}|$$

Actually, these *(sufficient)* conditions also guarantee that the solution depends Lipschitz continuously on the initial condition. i.e., if $\boldsymbol{y(x)}$ is the solution to the original problem and now $\hat{\boldsymbol{y}}(\boldsymbol{x_0})$ also satisfies the ODE but with a different initial condition $\exists$ a positive constant $\boldsymbol{K}$ such that $|\boldsymbol{y(x)} - \hat{\boldsymbol{y}}(\boldsymbol{x})| \leq \boldsymbol{K}\,|\,\boldsymbol{y(x_0)} - \hat{\boldsymbol{y}}(\boldsymbol{x_0})|$.

These conditions together with the existence and uniqueness of a solution defines a *well-posed* problem (2).

The following lemma will be useful for establishing the properties.

Lemma 3.0.2 *Let* $\boldsymbol{Let}\ \{\boldsymbol{\delta_i i = 0(1)n}\}$ *be set of real numbers. If there exist finite constants* $\boldsymbol{\Gamma}$ $\boldsymbol{and}$ $\boldsymbol{\Pi}$ $\boldsymbol{such\ that}$

$$|\boldsymbol{\delta_{i+1}}| \leq \boldsymbol{\Gamma}|\boldsymbol{e_i}| + \boldsymbol{\Pi}, \boldsymbol{i = 0\ (1) n - 1}, \tag{7}$$

then

$|\delta_i| \leq \frac{\Gamma^i - 1}{\Gamma - 1}\,\Pi + \Gamma^i|e_0|, \Gamma \neq 1.$ (8)

Proof. When $i = 0$, (8) is satisfied identically as $|e_0| \leq |e_0|$.
Suppose (8) holds whenever $i \leq j$ so that

$|\delta_j| \leq \frac{\Gamma^j - 1}{\Gamma - 1}\,\Pi + \Gamma^j|e_0|.$ (9)

Then, from (7) $i \leq j$ implies that

$|\delta_{j+1}| \leq \Gamma|e_j| + \Pi.$ (10)

On substituting (9) into (10), we have

$|\delta_{j+1}| \leq \frac{\Gamma^{j+1} - 1}{\Gamma - 1}\,\Pi + \Gamma^{j+1}|e_0|.$ (11)

Hence, (8) holds for all $i \geq 0$

3.1 Stability

The following theorem guarantees the stability of the newly proposed *MIME* and other *Euler methods* (2) – (6)

Theorem 3.1.1 *Suppose the IVP (1) satisfies the hypotheses of theorem (3.0.1), then the methods are stable.*

Proof Let y_n *and* z_n be two sets of solutions generated recursively by the *Euler methods* with the initial condition

$$y(x_0) = y_0, z(x_0) = z_0, |y_0 - z_0| = \delta_0$$

Let

$\delta_n = y_n - z_n, n \geq 0,$ (12)

and

$y_{n+1} = y_n + h\Phi(x_n, y_n; h),$ (13)

$z_{n+1} = z_n + h\Phi(x_n, z_n; h)$ (14)

This implies that

$y_{n+1} - z_{n+1} = y_n - z_n + h\{\Phi\,(x_n, y_n; h) - \Phi(x_n, z_n; h)\},$ (15)

And for the individual methods, we have

$$y_{n+1} - z_{n+1} = \begin{cases} y_n - z_n + h\{\Phi_{EM}(x_n, y_n; h) - \Phi_{EM}(x_n, z_n; h)\} \\ y_n - z_n + h\{\Phi_{ME}(x_n, y_n; h) - \Phi_{ME}(x_n, z_n; h)\} \\ y_n - z_n + h\{\Phi_{IE}(x_n, y_n; h) - \Phi_{IE}(x_n, z_n; h)\} \\ y_n - z_n + h\{\Phi_{IME}(x_n, y_n; h) - \Phi_{IE}(x_n, z_n; h)\} \\ y_n - z_n + h\{\Phi_{MIME}(x_n, y_n; h) - \Phi_{MIME}(x_n, z_n; h)\} \end{cases} \quad (16)$$

Using (12) and triangle inequality, we have :

$EM: |\delta_{n+1}| \leq (1 + hL)|\delta_n|, \ n \geq 0$ (17)

$ME: |\delta_{n+1}| \leq (1 + hL)|\delta_n|, \ n \geq 0$ (18)

$IE: |\delta_{n+1}| \leq (1 + hL)|\delta_n|, \ n \geq 0$ (19)

$IME: |\delta_{n+1}| \leq (1 + hL)|\delta_n|, \ n \geq 0$ (19)

$MIME: |\delta_{n+1}| \leq (1 + hL)|\delta_n|, \ n \geq 0$ (20)

If we assume $\Gamma = 1 + hL$, **and** $\Pi = 0$, then Lemma 3.0.2 implies that

$|\delta_n| \leq K|\delta_0|,$ (22)

where

$$K = e^{L(b-a)} < \alpha,$$

which implies the stability of the newly proposed *Modified Improved Modified Euler* and other *Euler method*

3.2 Absolute Stability

A special stability concept is that of absolute stability, which is normally associated with inherently stable IVPs (such as (1) whose Jacobians $(\frac{\partial f}{\partial y})$ have eigenvalues with negative real parts [6, 7]). If the step length used for the implementation of a numerical method is too small, excessive computation time and round--off error result. We should also consider the opposite case, and ask whether there is any upper bound on step length. Often there is such a bound, and it is reached when the method becomes numerically unstable: the numerical solution produced no longer corresponds qualitatively with the exact solution because some bifurcation has occurred. In this section, we study the absolute stability of the newly proposed (MIME) and other *Euler methods* (2) - (6).

Absolute stability analysis of one-step methods is usually carried out using the linear model problem

$$\boldsymbol{y' = \lambda y, y(x_0) = y_0, x_0 \le x} \tag{23}$$

where $\boldsymbol{\lambda}$ is complex. This has the analytical solution

$$\boldsymbol{y(x) = \eta e^{\lambda(x - x_0)}} \tag{24}$$

The problem has a stable fixed point at $\boldsymbol{y = 0 \; for \; Re(\lambda)} < 0$.
The region of absolute stability for a method is then the set of values of □ (real and non-negative) and $\boldsymbol{\lambda}$ (complex) for which $\boldsymbol{y_n \to 0 \; as \; n \to} \infty$, that is, for which the fixed point at the origin is stable. Thus, we want the set of values of $\boldsymbol{h \; and \; \lambda}$ for which $|\boldsymbol{R_{Method}(h\lambda)}| \le \boldsymbol{1}|$ **where** $\boldsymbol{R_{Method}(h\lambda)} \le \boldsymbol{1}$, the stability function, is the eigenvalue of the Jacobian of the Euler methods map evaluated at the fixed point.

Using the linear model problem (23), the Euler methods (2) – (6) give the following recurrence relation

$$\boldsymbol{EM: \; y_{n+1} = (1 + h\lambda)y_n} \tag{25}$$

$$\boldsymbol{ME: \; y_{n+1} = \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2\right)y_n} \tag{26}$$

$$\boldsymbol{IE: \; y_{n+1} = \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{4}h^3\lambda^3\right)y_n} \tag{27}$$

$$\boldsymbol{IME: \; y_{n+1} = \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{2}h^3\lambda^3\right)y_n} \tag{28}$$

$$\boldsymbol{MIME: \; y_{n+1} = \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{4}h^3\lambda^3\right)y_n} \tag{29}$$

Using the new *MIME* as a case study, the solution which satisfies the initial condition $\boldsymbol{y_0 = 1}$ is

$$\boldsymbol{y_n = \lim_{h \to 0} \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{4}h^3\lambda^3\right)^n} \tag{30}$$

In order to examine the convergence of this at, say, $\boldsymbol{x_n}$ it is necessary to study the behaviour of this function as $\boldsymbol{h}$ tends to zero in such a manner that $\boldsymbol{x_n}$ remains fixed.
Now

$$\boldsymbol{y_n = \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{4}h^3\lambda^3\right)^{\frac{x_n}{h}}} \tag{31}$$

so that

$$\ln y_n = \frac{x_n}{h}\ln\left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{4}h^3\lambda^3\right) \tag{32}$$

Then by de l'Hopital's rule

$$= \lim_{h\to 0}\frac{1}{h}\ln\left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{4}h^3\lambda^3\right) \tag{33}$$

$$= \lambda \tag{34}$$

Hence,

$$\lim_{h\to 0}\ln y_n = \lambda x_n \tag{35}$$

and thus,

$$\lim_{h\to 0} y_n = e^{\lambda x_n} \tag{36}$$

Then the method is consistent and convergent to $O(h^3)$ If we let $z = h\lambda$ then, the general form of the stability function of *MIME method is*:

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{4}z^3 \tag{37}$$

The stability functions of the new method show that it is a second-order method. Similarly, the stability function and order of convergence of other Euler methods are displayed in the table 1

Table 1: Stability functions and Region of Absolute Stability of *MIME and other Euler Methods*

Method	$R_{Method}(z)$	Stability Function	Region of Absolute Stability
EM	$R_{EM}(z)$	$1 + z$	$-2 < z < 0$
ME	$R_{ME}(z)$	$1 + z + \frac{1}{2}z^2$	$-2 < z < 0$
IE	$R_{IE}(z)$	$1 + z + \frac{1}{2}z^2 + \frac{1}{4}z^3$	$-2 < z < 0$
IME	$R_{IME}(z)$	$1 + z + \frac{1}{2}z^2 + \frac{1}{2}z^3$	$-1.47797 < z < 0$
MIME	$R_{MIME}(z)$	$1 + z + \frac{1}{2}z^2 + \frac{1}{4}z^3$	$-2 < z < 0$

3.3 Convergence

For a difference approximation to be usable for a class of functions $f(x_n, y\,(x_n))$ it is necessary that any function in this class satisfies a number of requirements as mentioned earlier on [7]. One of such requirement is the convergence of the method. Though convergence is implied by the consistency condition proved above. However, a succinct overview of the test of convergence is presented below:

Lemme 3.3.1 *Suppose the IVP (1) satisfies the hypothesis of the Existence and Uniqueness Theorem (3.0.1), and then the increment function Φ_{MIME} specified by (6) satisfies a Lipschitz condition of order 2 with respect to the independent variable y.*

Proof. Suppose *L* is the Lipschitz constant for $f\,(x, y)$ w.r.t. *y*, then, by theorem (3.0.1)

$$|f(x,y) - f(x,\hat{y})| \le L\,|y - \hat{y}|$$

Using (6),

$$|\Phi_{MIME}\,(x_n, y_n; h) - \Phi_{MIME}\,(x_n, z_n; h)|$$

$$= \left|hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right)\right)\right.$$
$$\left. - hf\left(x_n + \frac{1}{2}h, z_n + \frac{1}{2}hf\left(x_n + \frac{1}{2}h, z_n + \frac{1}{2}hf(x_n, y_n)\right)\right)\right|$$

$$< L|y_n - z_n| \left\{h + \tfrac{1}{2}h^2 + \tfrac{1}{4}h^3\right\}$$
$$< L^* \, |y_n - z_n|$$

where the Lipschitz constant $L*$ is given as

$$L^* = L\left\{h + \tfrac{1}{2}h^2 + \tfrac{1}{4}h^3\right\} \qquad (39)$$

Thus, the proposed method is convergent and it's order of accuracy is 2.

4. Numerical Computations

In this section, we discuss the implementation of the proposed *MIME and existing Euler Methods* on the IVP given by

$$y'(x) = -10\left((y(x^2 - 1), y\,(0)\right) = 2 \qquad (39)$$

Four numerical experiments comprising of 1000 steps each, were performed on this IVP as follows:

A. for $x = 0\ (0.01)10\ i.e\ h = 0.01$
B. for $x = 0\ (0.02)20\ i.e\ h = 0.02$
C. for $x = 0\ (0.03)30\ i.e\ h = 0.03$
D. for $x = 0\ (0.04)40\ i.e\ h = 0.04$

The numerical results of the experiments **A – D** are displayed in tables 2 – 5 and figure 1 – 4. In order to distinguish between the methods in the graphs plotted for the numerical values, we limited the display to $0.1 \le x \le 0.23$, though the problem was solved for $x = 0 \to 10, 20, 30, 40$ respectively.

Table 2: Numerical Values and Absolute Error of y(x) for x = 0 (0.01) 10

h = 0.01		Numerical Values of y(x)					Absolute Error of the Numerical Values of y(x)				
x	yExact	EM	ME	IE	IME	MIME	EM	ME	IE	IME	MIME
0	2	2	2	2	2	2	0	0	0	0	0
1	1.090909	1.088913	1.090971	1.090949	1.090804	1.090886	0.001995996	6.19185E-05	3.96068E-05	0.000105313	2.32462E-05
2	1.047619	1.046924	1.047637	1.04763	1.047589	1.047612	0.000695061	1.77296E-05	1.13597E-05	3.01534E-05	6.6311E-06
3	1.032258	1.031898	1.032266	1.032263	1.032244	1.032255	0.000359869	8.25589E-06	5.29282E-06	1.40413E-05	3.08371E-06
4	1.02439	1.024168	1.024395	1.024393	1.024382	1.024388	0.000222536	4.75476E-06	3.04918E-06	8.08684E-06	1.77478E-06
5	1.019608	1.019456	1.019611	1.01961	1.019603	1.019607	0.000152302	3.08668E-06	1.97982E-06	5.24985E-06	1.15168E-06
6	1.016393	1.016282	1.016396	1.016395	1.01639	1.016393	0.000111322	2.16406E-06	1.38821E-06	3.68067E-06	8.0721E-07
7	1.014085	1.013999	1.014086	1.014086	1.014082	1.014084	8.52142E-05	1.60081E-06	1.02699E-06	2.7227E-06	5.96996E-07
8	1.012346	1.012278	1.012347	1.012346	1.012344	1.012345	6.75007E-05	1.23192E-06	7.90383E-07	2.0953E-06	4.59357E-07
9	1.010989	1.010934	1.01099	1.01099	1.010987	1.010989	5.48997E-05	9.77267E-07	6.27033E-07	1.66218E-06	3.64359E-07
10	1.009901	1.009855	1.009902	1.009901	1.0099	1.009901	4.55982E-05	7.94123E-07	5.09546E-07	1.35068E-06	2.9605E-07

Table 3: Numerical Values and Absolute Error of y(x) for x = 0 (0.02) 20

h = 0.02		Numerical Values of y(x)					Absolute Error of the Numerical Values of y(x)				
x	yExact	EM	ME	IE	IME	MIME	EM	ME	IE	IME	MIME
0	2	2	2	2	2	2	0	0	0	0	0
2	1.047619	1.046216	1.047697	1.047667	1.047486	1.047587	0.001402791	7.80513E-05	4.75286E-05	0.000133411	3.22899E-05
4	1.02439	1.023941	1.024411	1.024403	1.024354	1.024382	0.000449276	2.08812E-05	1.27418E-05	3.57445E-05	8.61583E-06
6	1.016393	1.016169	1.016403	1.016399	1.016377	1.01639	0.000224788	9.49575E-06	5.79849E-06	1.62636E-05	3.91456E-06
8	1.012346	1.012209	1.012351	1.012349	1.012336	1.012343	0.000136311	5.40327E-06	3.30065E-06	9.25694E-06	2.22647E-06
10	1.009901	1.009809	1.009904	1.009903	1.009895	1.0099	9.20823E-05	3.48216E-06	2.12758E-06	5.9667E-06	1.43447E-06
12	1.008264	1.008198	1.008267	1.008266	1.00826	1.008263	6.66709E-05	2.4294E-06	1.48457E-06	4.16327E-06	1.0006E-06
14	1.007092	1.007042	1.007094	1.007093	1.007089	1.007091	5.0664E-05	1.79078E-06	1.09444E-06	3.06913E-06	7.37482E-07
16	1.006211	1.006171	1.006213	1.006212	1.006209	1.006211	3.98987E-05	1.37448E-06	8.4008E-07	2.35581E-06	5.65986E-07
18	1.005525	1.005493	1.005526	1.005526	1.005523	1.005524	3.22943E-05	1.08812E-06	6.65093E-07	1.86508E-06	4.48032E-07
20	1.004975	1.004948	1.004976	1.004976	1.004974	1.004975	2.67138E-05	8.8274E-07	5.39588E-07	1.51311E-06	3.63447E-07

Table 4: Numerical Values and Absolute Error of y(x) for x = 0 (0.03) 30

	0.03	Numerical Values of y(x)					Absolute Error of the Numerical Values of y(x)				
x	yExact	EM	ME	IE	IME	MIME	EM	ME	IE	IME	MIME
0	2	2	2	2	2	2	0	0	0	0	0
3	1.032258	1.031154	1.032348	1.03231	1.032105	1.032217	0.001103571	9.00061E-05	5.15154E-05	0.00015327	4.06054E-05
6	1.016393	1.016052	1.016417	1.016407	1.016353	1.016383	0.000341492	2.35069E-05	1.34882E-05	4.01583E-05	1.05918E-05
9	1.010989	1.010821	1.011	1.010995	1.010971	1.010984	0.000168406	1.06022E-05	6.08871E-06	1.81329E-05	4.77527E-06
12	1.008264	1.008163	1.00827	1.008268	1.008254	1.008262	0.00010126	6.00785E-06	3.45174E-06	1.02812E-05	2.70543E-06
15	1.006623	1.006555	1.006626	1.006625	1.006616	1.006621	6.8013E-05	3.8621E-06	2.2195E-06	6.61151E-06	1.73896E-06
18	1.005525	1.005476	1.005528	1.005526	1.00552	1.005524	4.90358E-05	2.68996E-06	1.54615E-06	4.60601E-06	1.21109E-06
21	1.004739	1.004702	1.004741	1.00474	1.004736	1.004738	3.71401E-05	1.98048E-06	1.1385E-06	3.39175E-06	8.91615E-07
24	1.004149	1.00412	1.004151	1.00415	1.004147	1.004149	2.91704E-05	1.51871E-06	8.73127E-07	2.60126E-06	6.83698E-07
27	1.00369	1.003666	1.003691	1.003691	1.003688	1.003689	2.35583E-05	1.20145E-06	6.90781E-07	2.05806E-06	5.40855E-07
30	1.003322	1.003303	1.003323	1.003323	1.003321	1.003322	1.94505E-05	9.74138E-07	5.60118E-07	1.66881E-06	4.38514E-07

Table 5: Numerical Values and Absolute Error of y(x) for x = 0 (0.04) 40

h = 0.04		Numerical Values of y(x)					Absolute Error of the Numerical Values of y(x)				
x	yExact	EM	ME	IE	IME	MIME	EM	ME	IE	IME	MIME
0	2	2	2	2	2	2	0	0	0	0	0
4	1.02439	1.023463	1.024492	1.024444	1.024218	1.024341	0.000927076	0.000101363	5.37268E-05	0.000172049	4.95529E-05
8	1.012346	1.012065	1.012372	1.01236	1.012301	1.012333	0.000281057	2.61425E-05	1.38992E-05	4.46014E-05	1.2784E-05
12	1.008264	1.008127	1.008276	1.008271	1.008244	1.008259	0.000137376	1.17408E-05	6.24873E-06	2.00665E-05	5.74208E-06
16	1.006211	1.006129	1.006218	1.006215	1.0062	1.006208	8.21649E-05	6.63883E-06	3.53519E-06	1.13569E-05	3.24707E-06
20	1.004975	1.00492	1.004979	1.004977	1.004968	1.004973	5.49862E-05	4.26222E-06	2.27035E-06	7.29526E-06	2.08474E-06
24	1.004149	1.00411	1.004152	1.004151	1.004144	1.004148	3.95358E-05	2.96609E-06	1.58027E-06	5.07864E-06	1.45082E-06
28	1.003559	1.003529	1.003561	1.00356	1.003555	1.003558	2.98806E-05	2.18243E-06	1.16293E-06	3.73782E-06	1.06752E-06
32	1.003115	1.003092	1.003117	1.003116	1.003112	1.003114	2.34276E-05	1.6728E-06	8.91471E-07	2.86555E-06	8.18254E-07
36	1.00277	1.002751	1.002771	1.002771	1.002768	1.002769	1.88927E-05	1.32288E-06	7.0505E-07	2.26647E-06	6.47095E-07
40	1.002494	1.002478	1.002495	1.002494	1.002492	1.002493	1.55788E-05	1.07228E-06	5.71531E-07	1.83735E-06	5.24519E-07

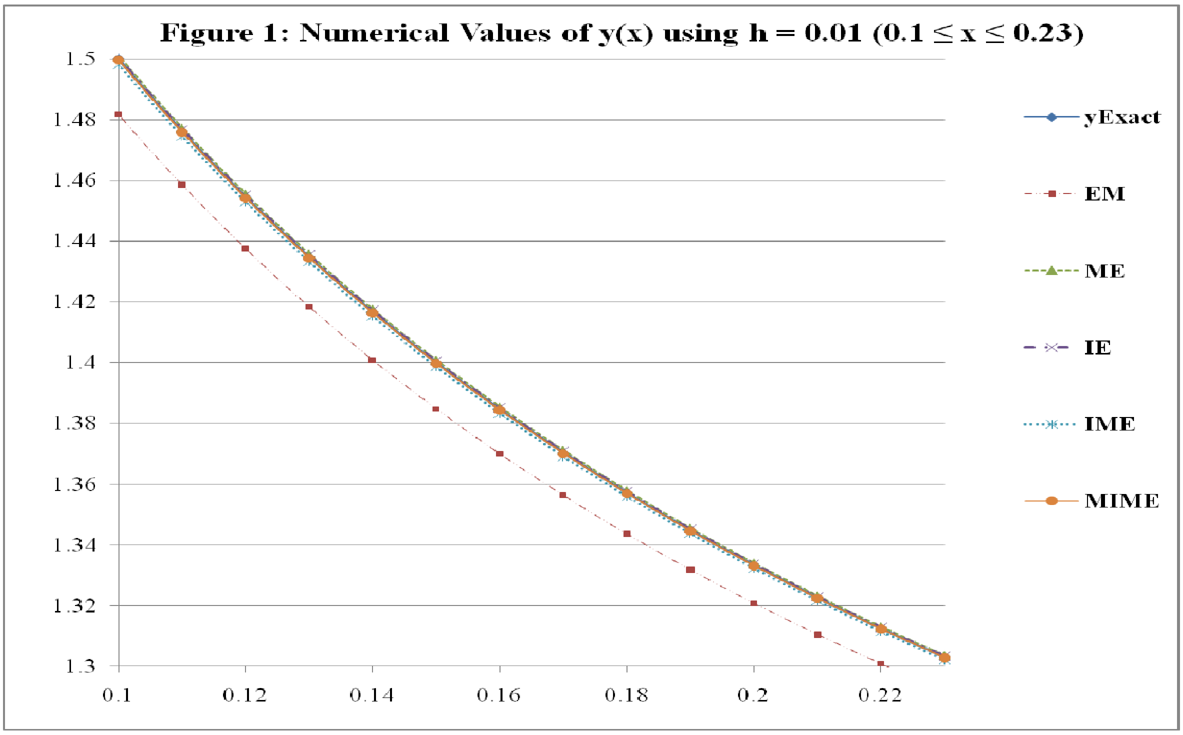


Figure 1: Numerical Values of y(x) using h = 0.01 (0.1 ≤ x ≤ 0.23)
1.5
1.48
1.46
1.44
1.42
1.4
1.38
1.36
1.34
1.32
1.3
0.1
0.12
0.14
0.16
0.18
0.2
0.22
yExact
EM
ME
IE
IME
MIME

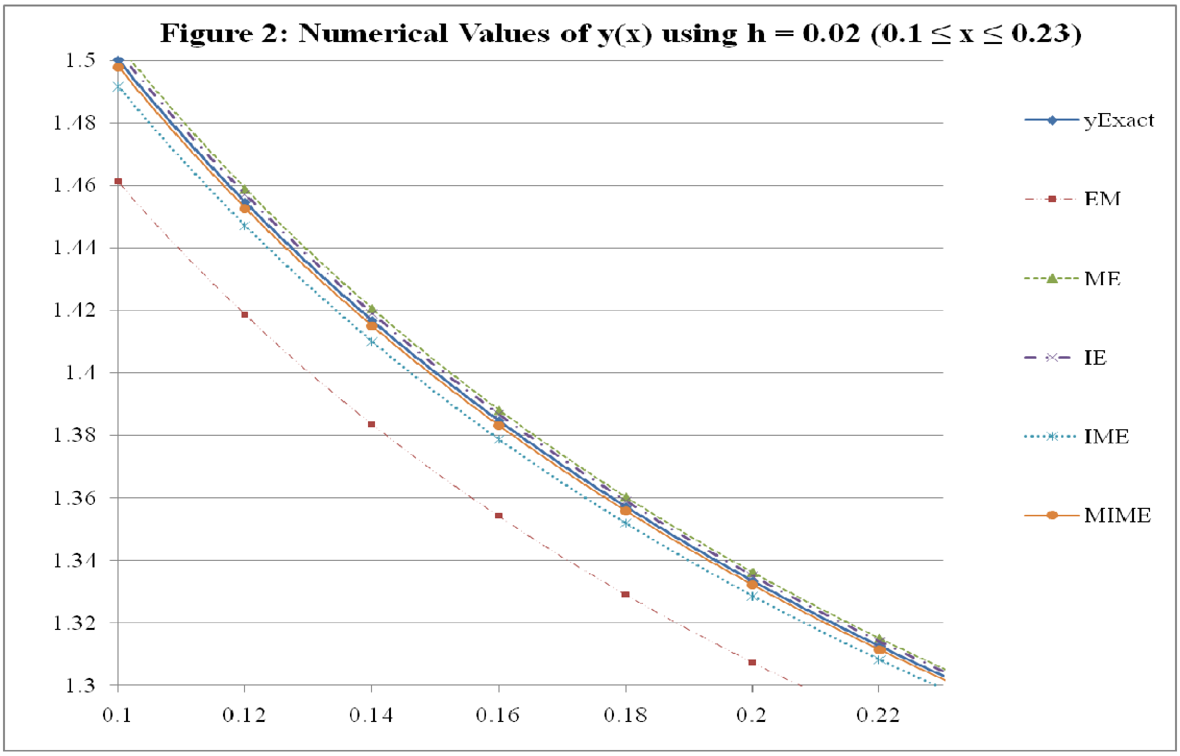


Figure 2: Numerical Values of y(x) using h = 0.02 (0.1 ≤ x ≤ 0.23)
1.5
1.48
1.46
1.44
1.42
1.4
1.38
1.36
1.34
1.32
1.3
0.1
0.12
0.14
0.16
0.18
0.2
0.22
yExact
EM
ME
IE
IME
MIME

Figure 3: Numerical Values of y(x) using h = 0.03 ($0.1 \le x \le 0.23$)

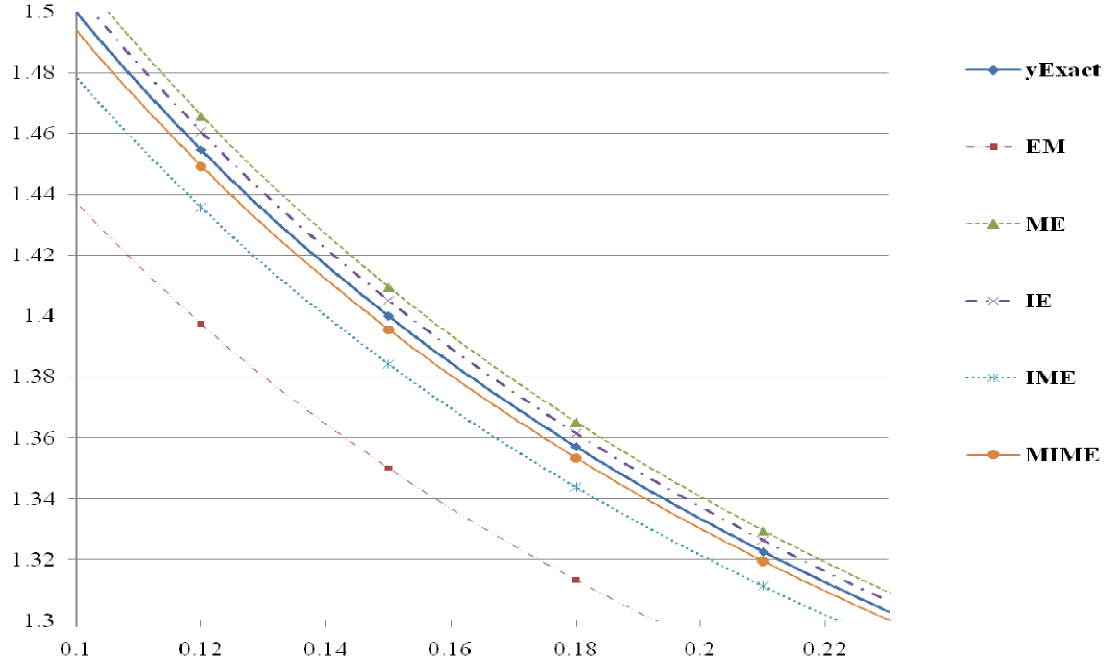
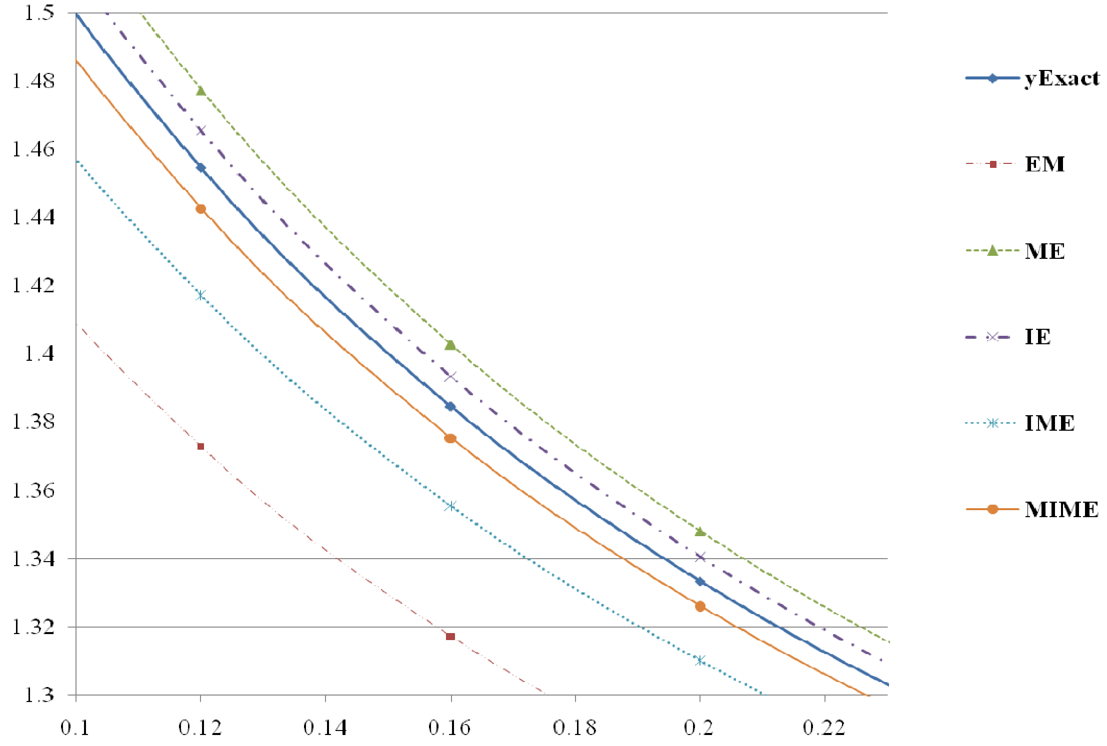


Figure 4: Numerical Values of y(x) using h = 0.04 ($0.1 \le x \le 0.23$)



CONCLUSION

The basic selection properties of a numerical method were analyzed in respect of the proposed *MIME and other existing Euler Methods*. This analysis of the stability, convergence and absolute stability show that the improvement that led to the development of the *Modified Improved*

Modified Euler Method is worthwhile. The comparison between the numerical values generated by this method with the theoretical solution show that, indeed the new scheme is accurate and efficient. From the numerical experiments the comparison of the results generated with those of the existing methods also show that the new method is the best as far as our computation is concerned.

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