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On irreducible characters of dihedral groups of degree 2^n $(n \geq 3)$

M. A. Mohammed and Idris M. Daya

Department of Mathematics, School of Science Education, Federal College of Education (Technical) Potiskum, Yobe State, Nigeria

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ABSTRACT

In this paper, the irreducible characters of dihedral groups of order 2^n $(n \geq 3)$ *were discussed. The method adopted uses some basic facts on dihedral groups, representation theory and the Group's Algorithms and Programming (GAP) to obtain the results. It was observed that all dihedral groups of order* 2^n $(n \geq 3)$ *are non abelian and non simple groups.*

Key words: Dihedral groups, Simple, Abelian, Irreducible, Character

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INTRODUCTION

A dihedral group is a group of symmetries of a regular polygon, including both the rotation and reflection operations. Dihedral groups are good example of finite groups and have a series of applications in Chemistry, Physics, Mathematical Sciences and Engineering.

The dihedral group $D_n$is that group generated by two elements, $x and y$, which satisfy the relation, $x^n = y^2; (xy)^2 = e$.

In this paper, $D_n$refers to the symmetry of a regular polygon with $n$sides.

Conventionally, we write

$D_n = \{1, x, x^2, \dots \dots x^{n-1}, y, xy, x^2y, \dots \dots x^{n-1}y\},$

In section two, some basic concepts of representation theory were discussed. Section three, dwells on the application of The Groups Algorithms and Programming (GAP) version 4.12 to discuss the Commutativity and Simplicity of dihedral groups of degree2^n $(n \geq 3)$. The main results of this paper will be stated in section four.

MATERIALS AND METHODS

To begin with, there is the need for some preliminary fact and brief discussion of notations.

2.1 Definition
Let $\rho: G \to GL(n, F)$be a representationof a group G over a field F. The function $\chi: G \to F$ defined by $\chi(g) = tr(\rho(g))$is called character of ρ.

2.2 Definition (Feit 1971)
An irreducible character of degree one, is called a linear character.

2.3 Corollary
If $\chi_1, \dots \dots \dots, \chi_k$are all the irreducible characters of G and $\chi = \sum_{i=1}^{k} n_i \chi_i$and $\phi = \sum_{i=1}^{k} m_i \chi_i$are any two characters of G, then $\langle \chi, \phi \rangle = \sum_{i=1}^{k} n_i m_i$.

Proof:
$\langle \chi, \phi \rangle = \langle \sum_{i=1}^{k} n_i \chi_i, \sum_{i=1}^{k} m \chi_j \rangle = \sum_{i=1}^{k} n_i m_j \langle \chi_i, \chi_j \rangle = \sum_{i=1}^{k} n_i m_i$.Thus$\langle \chi, \phi \rangle \geq 0$ and $\langle \chi, \phi \rangle \in \mathbb{Z}$and in particular, $\langle \chi, \chi \rangle = \sum_{i=1}^{k} n_i^2$

2.4 Definition
A representation ρ of $G$is said to be irreducible if it is not a direct sum of other representation of G.Also a character is said to be irreducible if it is not a sum of other characters of G.

2.5 Theorem
Let $\mathbb{F} = \mathbb{C}$.If G is finite Abelian group then every irreducible representation of G is linear.

2.6 Theorem (Maschke's Theorem)
Let G be a finite group and $\mathbb{F}$ a field whose characteristics is either 0 or a prime that does not divide $|G|$, then every representation of G over $\mathbb{F}$ is completely reducible.

2.7 Lemma
Suppose $\mathbb{F} = \mathbb{C}$and let χbe character of G. Then $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$, where $\overline{\chi(g)}$ denote the complex conjugate $\chi(g)$.

Proof:
Let χbe the character of the representation ρ of G of degree n and let $H = \langle g \rangle$be a cyclic subgroup of G generated by the element $g \in G$. We can certainly restrict $\rho$to a representation S of H. By Maschke's theorem and Lemma 2.4 we know that S is completely reducible and hence direct sum of the irreducible representations. From Theorem 2.5 we also know that all irreducible representation of H have degree one, since H is abelian. Thus we have for all $g \in G$:

$$S(g) \sim \begin{pmatrix} \zeta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \zeta_k \end{pmatrix},$$

Where each $\zeta_i \in \mathbb{C}$, $i = 1, \dots \dots k$. If $|g| = m$, then $S(g^m) = S(g)^m = I$ and hence
${\zeta_i}^m = 1 \; \forall \; i$. Thus $|\zeta_i| = 1$and it follows that $\zeta_i^{-1} = \overline{\zeta_i}$.We get

$$S(g^{-1}) = S(g)^{-1} \sim \begin{pmatrix} {\zeta_1}^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & {\zeta_K}^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\zeta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \overline{\zeta_k} \end{pmatrix}.$$

Hence $\chi(g^{-1}) = \overline{\zeta_1} + \cdots \dots . . + \overline{\zeta_k} = \overline{\zeta_1 + \cdots \dots \dots . + \zeta_k} = \overline{\chi(g)}$.

2.8Lemma
Let χbe character of G. Then $\chi$is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

Proof: [Leder, 1977]
($\Rightarrow$) Let C be defined as before, but assume now that $T \cong T'$. We derive from $T(g)C = CT(g)$and from Schur's Lemma that $C = \lambda I$, $\lambda \in \mathbb{F}$. On one hand we have $tr(C) = \lambda n$ and so on the other hand
$tr(C) = |G| tr(M)$, Since$tr(T(k)^{-1} M T(k) = tr(M) \forall k \in G$.

Hence $\lambda = \frac{|G|}{n} tr(M)$. We equate the two expressions for C to get

$$\sum_{k\in G} T(k^{-1})MT(k) = \lambda I = \frac{|G|}{n} tr(M)I = \frac{|G|}{n}(m_1 \dots \dots \dots \dots m_{nn})I$$

Again this must hold for any arbitrarily defined matrix M and by equating coefficient in the above equation we derive:

$$\sum_{k\in G} t_{ij}(k^{-1})t'_{pq}(k) = \begin{cases} 0, & if i \neq q or j \neq p \\ \frac{|G|}{n}, & if i = q and j = p \end{cases}$$

If we let $\delta_{ij} = \begin{cases} 1 & if i = j \\ 0, & if i \neq j \end{cases}$, then we rephrase the above as

$$\langle t_{ij}, t_{pq} \rangle = \frac{1}{n}\delta_{iq}\delta_{jp}.$$

We conclude that

$$\langle \chi, \chi \rangle = \sum_{i,j} \langle t_{ii}, t_{jj} \rangle = \sum_{i,j} \frac{1}{n}\delta_{ij}\delta_{ij} = 1$$

($\Leftarrow$) If $\chi_1 \dots \dots \dots . \chi_k$ are all irreducible characters of G, we can express χ as $\chi = \sum_{i=1}^{k} n_i \chi_i n_i \geq 0$. Assume that $\langle \chi, \chi \rangle = 1$, the by corollary 2.3

$$\langle \chi, \chi \rangle = \sum_{i=1}^{k} n_i^2 = 1.$$

Since $n_i \in \mathbb{Z}$ and $n_i \geq 0$, we have that for one i, $n_i = 1$ and for all $j \neq i$, $n_j = 0$, and so $\chi = \chi_i$ and χ is irreducible.

2.9Lemma

If $\chi_1 \dots \dots \dots . \chi_k$ are all the irreducible characters of G, then $\sum_{i=1}^{k} {\chi_i}^2(1) = |G|$.

Proof:

$$|G| = \chi_R(1) = \left[\sum_{i=1}^{k} \chi_i(1)\chi_i\right](1) = \sum_{i=1}^{k} {\chi_i}^2(1).$$

2.10Theorem

A group G is abelian if and only if every irreducible character χ_i of G is linear.

Proof:

($\Rightarrow$) Proved in Theorem 2.7

($\Leftarrow$) Let $\chi_1 \dots \dots , \chi_k$ be all the irreducible characters of G. Assume that, $\forall i = 1, \dots \dots , k$, $\chi_i(1) = 1$. Then $\sum_{i=1}^{k} |\chi_i(1)|^2 = k$.

From Lemma 2.9 we also have $\sum_{i=1}^{k} |\chi_i(1)|^2 = |G|$.

Hence $|G| = k$ and every element of G forms its own conjugacy class. Consequently, from standard results in group theory, [Green 1988, p. 78], we can deduce that all element of G lies in the centre of G. Thus G is abelian.

2.11Corollary

If $H \lhd G$, then $H = \cap \{ ker\chi_i \mid H \leq ker\chi_i \}$.

2.12Corollary

Let χ_1 be the character of the trivial representation. The G is simple if and only if $ker\chi_i = 1 \;\; for \;\; 2 \leq i \leq k$.

Proof:
(⇒) For $i = 1, ker\chi_1 = G$. Suppose G is simple, i.e. G does not have any proper normal subgroup. To reach a contradiction, assume $ker\chi_i = N, N \neq 1$, for some $2 \leq i \leq k$. But the kernel of $\chi_1$is a normal subgroup of G, since $\chi_1$is a linear mapping from G to$\mathbb{C}$. Hence the required contradiction.

(⇐) Assume $ker\chi_i = 1 \ for \ 2 \leq i \leq k$.To reach a contradiction suppose $H \lhd G$ for $H \neq 1$. There exists at least one representation such that $H \leq ker\chi_i$, i.e. the trivial representation. Then from Corollary 2.7 we have $H = \cap \{ker\chi_i \mid H \leq ker\chi_i\}$and the contradiction follows

3. 0 Dihedral Groups of Degree $2^n$for $n \geq 3$.
We shall now determine using Groups, Algorithms and Programming (**GAP**) some Dihedral groups of Degree 2^n ($n \geq 3$) (4,8,16,32,64,128,256,512,1024,2048,40968192, 16384,32768, 65536, 131072, 262144, 524288 and 1048576) and discuss whether they are Simple or Abelian which will guide us to obtain our result.

3.1 Groups, Algorithms and Programming GAP
gap># The DihedralGroup of symmetry, D_4

```
gap> D4:=DihedralGroup(IsGroup,8); Group([(1,2,3,4),(2,4)])
D4={(1),(1,3)(2,4),(1,4,3,2),(1,2,3,4),(2,4),(1,3),(1,4)(2,3),(1,2)(3,4)}
gap> D4:=DihedralGroup(IsGroup,8);Group([ (1,2,3,4), (2,4) ])
gap> for i in D4 do
> Print(i,"\n");
> od;
(1),(1,3)(2,4),(1,4,3,2),(1,2,3,4),(2,4),(1,3),(1,4)(2,3),(1,2)(3,4)
gap>C1:=CharacterTable(D4);CharacterTable(Group([(1,2,3,4),(2,4) ]))
Gap> Display(C1,rec(Powermap:=false,Centralizers:=false));C1
     1a  2a  2b  4a  2c
X.1   1   1   1   1   1
X.2   1  -1  -1   1   1
X.3   1  -1   1  -1   1
X.4   1   1  -1  -1   1
X.5   2   .   .   .  -2
gap> IsAbelian(C1);false
gap> IsSimple(C1);false
```

gap># The DihedralGroup of symmetry, D_8

```
gap> D8:=DihedralGroup(IsGroup,16);
Group([(1,2,3,4,5,6,7,8),(2,8)(3,7)(4,6)])
gap> for i in D8 do
> Print(i,"\n");
> od;
(1),(1,5)(2,6)(3,7)(4,8),(1,7,5,3)(2,8,6,4),(1,3,5,7)(2,4,6,8),(1,8,7,6,5,4,3
,2),(1,4,7,2,5,8,3,6),(1,6,3,8,5,2,7,4),(1,2,3,4,5,6,7,8),(2,8)(3,7)(4,6),(1,
5)(2,4)(6,8),(1,7)(2,6)(3,5),(1,3)(4,8)(5,7),(1,8)(2,7)(3,6)(4,5),(1,4)(2,3)(
5,8)(6,7),(1,6)(2,5)(3,4)(7,8),(1,2)(3,8)(4,7)(5,6)
gap>C2:=CharacterTable(D8);CharacterTable(Group([(1,2,3,4,5,6,7,8),(2,8)(3,7)(
4,6)]))

gap>Display(C2,rec(Powermap:=false,Centralizers:=false)); C2
      1a  2a  2b  8a  4a  8b  2c
X.1   1   1   1   1   1   1   1
X.2   1  -1  -1   1   1   1   1
X.3   1  -1   1  -1   1  -1   1
X.4   1   1  -1  -1   1  -1   1
X.5   2   .   .   .  -2   .   2
X.6   2   .   .   A   .  -A  -2
X.7   2   .   .  -A   .   A  -2
```

A = -E(8)+E(8)^3 = -ER(2) = -R2

gap> IsSimple(C_2);false
gap> IsAbelian(C_2);false
gap># The DihedralGroup of symmetry, D_{16}
gap>D_{16}:=DihedralGroup(IsGroup,32);
Group([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16),(2,16)(3,15)(4,14)(5,13)(6,12)(7,11)(8,10)])
gap>C_3:=CharacterTable(D_{16});CharacterTable(Group([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16),(2,16)(3,15)(4,14)(5,13)(6,12)(7,11)(8,10)]))
gap> IsSimple(C_3);false
gap> IsAbelian(C_3);false

gap># The DihedralGroup of symmetry, D_{32}
gap>D_{32}:=DihedralGroup(IsGroup,64);
Group([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32),(2,32)(3,31)(4,30)(5,29)(6,28)(7,27)(8,26)(9,25)(10,24)(11,23)(12,22)(13,21)(14,20)(15,19)(16,18)])
gap> C_4:=CharacterTable(D_{32});
CharacterTable(Group([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32),(2,32)(3,31)(4,30)(5,29)(6,28)(7,27)(8,26)(9,25)(10,24)(11, 23)(12,22)(13,21)(14,20)(15,19)(16,18)]))
gap> IsSimple(C_4);false
gap> IsAbelian(C_4);false

gap># The DihedralGroup of symmetry, D_{64}
gap> D_{64}:=DihedralGroup(IsGroup,128);
<permutation group with 2 generators>
gap> C_5:=CharacterTable(D_{64});CharacterTable(<permutation group of size 128 with 2 generators>)
gap> IsSimple(C_5);false
gap> IsAbelian(C_5);false

gap># The DihedralGroup of symmetry, D_{128}
gap> D_{128}:=DihedralGroup(IsGroup,256);
<permutation group with 2 generators>
gap> C_6:=CharacterTable(D_{128});CharacterTable(<permutation group of size 256 with 2 generators>)
gap> IsSimple(C_6);false
gap> IsAbelian(C_6);false

gap># The DihedralGroup of symmetry, D_{256}
gap> D_{256}:=DihedralGroup(IsGroup,512);
<permutation group with 2 generators>
gap> C_7:=CharacterTable(D_{256});CharacterTable(<permutation group of size 512 with 2 generators>)
gap> IsSimple(C_7);false
gap> IsAbelian(C_7);false

gap># The DihedralGroup of symmetry, D_{512}
gap> D_{512}:=DihedralGroup(IsGroup,1024);
<permutation group with 2 generators>
gap> C_8:=CharacterTable(D_{512});CharacterTable(<permutation group of size 1024 with 2 generators>)
gap> IsSimple(C_8);false
gap> IsAbelian(C_8);false

gap># The DihedralGroup of symmetry, D_{1024}

```
gap> D1024:=DihedralGroup(IsGroup,2048);
<permutation group with 2 generators>
Gap> C9:=CharacterTable(D1024);CharacterTable( <permutation group of size 2048 with 2 generators> )
gap> IsSimple(C9);false
gap> IsAbelian(C9);false
```

gap># The DihedralGroup of symmetry, D_{2048}

```
gap> D2048:=DihedralGroup(IsGroup,4096);
<permutation group with 2 generators>
Gap> C10:=CharacterTable(D2048);CharacterTable(<permutation group of size 4096 with 2 generators>)
gap> IsSimple(C10);false
gap> IsAbelian(C10);false
```

gap># The DihedralGroup of symmetry, D_{4096}

```
gap> D4096:=DihedralGroup(IsGroup, 8192);
<permutation group with 2 generators>
gap> C11:=CharacterTable(D4096);
gap> IsSimple(C11);false
gap> IsAbelian(C11);false
```

gap># The DihedralGroup of symmetry, D_{8192}

```
gap> D8192:=DihedralGroup(IsGroup,16384);
<permutation group with 2 generators>
gap> C12:=CharacterTable(D8192);
gap> IsSimple(C12);false
gap> IsAbelian(C12);false
```

gap># The DihedralGroup of symmetry, D_{16284}

```
gap> D16384:=DihedralGroup(IsGroup,32768);
<permutation group with 2 generators>
gap> C13:=CharacterTable(D16384);
gap> IsSimple(C13);false
gap> IsAbelian(C13);false
```

gap># The DihedralGroup of symmetry, D_{32768}

```
gap> D32768:=DihedralGroup(IsGroup,65536);
<permutation group with 2 generators>
gap> C14:=CharacterTable(D32768);
gap> IsSimple(C14);false
gap> IsAbelian(C14);false
```

gap># The DihedralGroup of symmetry, D_{65526}

```
gap> D65536:=DihedralGroup(IsGroup,131072);
<permutation group with 2 generators>
gap> C15:=CharacterTable(D65536);
gap> IsSimple(C15);false
gap> IsAbelian(C15);false
```

gap># The DihedralGroup of symmetry, D_{131072}

```
gap> D131072:=DihedralGroup(IsGroup,262144);
<permutation group with 2 generators>
gap> C16:=CharacterTable(D131072);
gap> IsSimple(C16);false
gap> IsAbelian(C16);false
```

gap>\# The DihedralGroup of symmetry, D_{262144}

```
gap> D262144:=DihedralGroup(IsGroup,524288);
<permutation group with 2 generators>
gap> C17:=CharacterTable(D262144);
gap> IsSimple(C17);false
gap> IsAbelian(C17);false
```

gap>\# The DihedralGroup of symmetry, D_{524288}

```
gap> D524288:=DihedralGroup(IsGroup,1048576);
<permutation group with 2 generators>
gap> C18:=CharacterTable(D524288);
gap> IsSimple(C18);false
gap> IsAbelian(C18);false
```

RESULTS

Based on the results obtained from the GAP in section 3.1, which concern particularly on Commutativity and Simplicity of dihedral group of degree2^n, n a positive integer ≥ 3, we now state the following remarks:

Let $D_n$be dihedral groups of degree2^n, n a positive integer $\geq$ 3.Then,
i. $D_n$is not Abelian
ii. $D_n$is not Simple

CONCLUSION

The Groups, Algorithms and Programming (GAP), were used to study the property of some dihedral groups of order$2^n$for $n \geq 3$ via its character tables. It was observed that group of this order$2^n$for $n \geq 3$ are non-abelian simple groups.

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