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On Maximal Subgroups of a Group with Unique Order

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ABSTRACT

This paper involves an investigation on the maximal subgroups of the groups of order pq^r, a theorem has been stated and prove concerning the nature and behavior of the maximal subgroup of such groups. The group algorithm and programming (GAP) has been applied to enhance and validate the result.

Keywords: Element, Abelian, Permutation groups

INTRODUCTION

The concept of maximal subgroup is playing a vital role in the application of group series more especially in determining the solvability and nil potency of a group. This research has investigated the maximal subgroups of a permutation group of unique order and carries out research on the behavior of the maximal subgroup.

Definition 1.1

Let G be a group. A subgroup H of G is said to be a maximal subgroup of G if $H \neq G$ and there is no subgroup $K \in G$ such that H<K<G. Note that a maximal subgroup of G is not maximal among all subgroups of G, but only among all proper subgroups of G. For this reason, maximal subgroups are sometimes called maximal proper subgroups.

Similarly, a normal subgroup N of G is said to be a maximal normal subgroup of G if N ≠ G and there is no normal subgroup K of G, such that N<K<G.

Definition 1.2

A subgroup N of a group G is normal in G if the left and right cossets are the same, that is if $gH = Hg \ \forall g \in G$ and a subgroup H of G.

Definition 1.3

The factor group of the normal subgroup N in a group G written as *G/N* is the set of cosets of N in G.

Definition 1.4

The factor group of the normal subgroup N in a group G written as *G/N* is the set of cosets of N in G.

Definition 1.5 **[1]**

A solvable group is a group having a normal series such that each normal factor is abelian.

Definition 1.6 **[2]**

A group G is nilpotent if it has a normal series $G=G_0 \leq G_1 \leq G_2 \leq \ldots \leq G_n=(1)$ where $G_i/G_{i+1} \leq Z(G/G_{i+1})$. (centre of G/G_{i+1}).

Definition 1.7

A finite group is simple when its only normal subgroups are the trivial subgroup and the whole group.

Definition 1.8 [3]

A group is a p-group if the order is a prime or prime power.

Definition 1.9

Let G be a group, and let p be a prime number. A group of order P^k for some $k \geq 1$ is called a p-group.

Theorem 2.1 (Lagrange's theorem)

Let G be a finite group and $H \leq G$. Then $|H|$ divides $|G|$ and $|G:H| = \frac{|G|}{|H|}$.

Proof

Let G be a finite group and let $H \leq G$ such that $|H|$ $=m$ and $|G:H| = k$. For any $g \in G$, define a map $\varnothing_g : H \to gH$ by $\varnothing_g (h)=gh$. This map is clearly surjective.

Furthermore, for any distinct $h_1 \neq h_2 \in H$, we have that $gh_1 \neq gh_2$. Thus $\varnothing_g$ is a bijection,

so $|gH| = |H| = m$

It is easy to see that the set of left cossets of H in G form a partition of G. First note that for any $g \in G$, $g \in gH$.

Thus, $G \subseteq \bigcup_{g \in G} gH$ and clearly $\bigcup_{g \in G} gH \in G$. Therefore,

$$G = \bigcup_{g \in G} gH$$

To show that left cossets are disjoint, suppose that for distinct cossets $g_1H \neq g_2H \in G/H$.

There exists an element $x \in g_1H \cap g_2H$. Then there exist elements $h_1, h_2 \in H$ such that, $x=g_1h_1=g_2h_2$.

Therefore, $g_1=g_2h_2h_1^{-1}$, so for any $g_1h \in g_1H$,

$$g_1h=(g_2h_2h_1^{-1})h=g_2(h_2h_1^{-1}h) \in g_2H$$

Hence, $g_1H \in g_2H$. But we have seen that $|g_1H| = |g_2H|$, so $g_1H = g_2H$, which is a contradiction. Thus, the k left cosets of H in G are in fact disjoint and, hence, partition G. Since each has cardinality m, it follows that $|G| = km$. Therefore $|G|$ divides $|G|$ and $|G|$.

Example

If $|G|=14$ then the only possible orders for a subgroup are 1, 2, 7 and 14.

Theorem 2.2 (Cauchy's theorem for abelian groups)

Let A be a finite abelian group. If p is a prime number that divides its order, then A must have an element of order p.

Theorem 2.3 (Sylow's first theorem)

Let G be a finite group and p be a prime number such that its power by α is the largest power that will divide $|G|$ Then there exist at least one subgroup of order P^α. Such groups are called Sylow p-subgroups.

Theorem 2.4 (Sylow's second theorem)

Let n_p be the number of sylow p-subgroups of a finite group G. Then $n_{p \equiv} 1 \ mod$.

Theorem 2.5 (Sylow's third theorem)

Any two Sylow p-subgroups are conjugate.

Theorem 2.6 (Sylow's fourth theorem)

Any p-subgroup B is contained in a Sylow p-subgroup.

Example

Let G be a group of order 12 [4-6]. Then either G has a normal Sylow 3-subgroup, or else it's isomorphic to A_4.

Reason: 12=22 × 3. We know n_3 has to divide 2^2=4 and it also has to be congruent to 1 *mod* 3. So it can be either 1 or 4.

If n_3=1, then G has a normal Sylow 3-subgroup.

If n_3=4, then we know that the four Sylow 3-subgroups are acted on by G, by conjugation. Let's call the set $S=\{P_1,\ldots,P_4\}$. The action of G gives us a homomorphism $\varnothing : G \rightarrow S_4$.

We will first show that Ø is injective, then we will show that the image of $\varnothing$ is A_4. This will show that $G \cong im\varnothing = A_4$.

To show $\varnothing$ is injective, we need to show that ker $\varnothing$ =1.

Ker $\varnothing = \{g \in G : gP_i g^{-1} = P_i \forall P_i \in S\} = \bigcap_{i=1}^{4} N_G(P_i)$

We know that for eachi, $i, n_3 = (G : N_G(P_i)) = |G| = |N_G(P_i)|$, so we have here that,

$|N_G(P_i)|$=12/4=3. Since $P_i \leq N_G(P_i)$ and $|P|$ is also 3, it means $P_i = N_G(P_i)$. So in our case,

ker $\varnothing = \bigcap_{i=1}^{4} P_i$

The P_i's happen, in this case, to be distinct groups of prime order (their order is 3). A general and useful fact about distinct groups of the same prime order is that they can only intersect each other trivially. (Take for example two subgroups P_1 and P_2 of order p, then the subgroup $P_1 \cap P_2$ has o have order 1 or p. If it has order p then $P_1=P_2$, so if P_1 and P_2 are not the same subgroup $P_1 \cap P_2$ has to have order 1, i.e., it's the trivial subgroup.)

Applying this to our case we get ker $\varnothing$ =1. Therefore $\varnothing$ is injective, and $G \cong im\varnothing$

Now G has 4 subgroups, $P_1,\ldots,P_4$, of order 3. Each of these subgroups has two elements of order 3 and the identity element.

The two elements of order three have to be different for each P_i (since different P_i 's had only the identity element in common).

Therefore G contains 8 different elements of order 3.

Since G is isomorphic to im$\varnothing$, these 8 different elements of order 3 have to map to 8 different elements of order 3 in S_4. The only elements of order three in S_4 are 3-cycles. And 3-cycles are even permutations, so are elements in A_4.

So $A_4 \cap im\varnothing$ is a subgroup of both A_4 and im$\varnothing$ with at least 8 elements. But since both A_4 and im$\varnothing$ have 12 elements, this intersection subgroup has to also divide 12. The only factor of 12 that is greater than or equal to 8 is 12. So $A_4 \cap im\varnothing$ is a subgroup of both A_4 and im$\varnothing$ of size 12, and since A_4 and im$\varnothing$only have 12 elements anyway, it means $A_4 \cap im\varnothing = A_4$=im$\varnothing$.

Example

Let G be a group of order 351. Then G has a normal Sylow p-subgroup for some prime p dividing 351.

Reasoning: 351=3^3 × 13. So a Sylow 3-subgroup would have order 3^3=27, and a Sylow 13-subgroup would have order 13. Let us start out with what n_{13} can be n_{13} divides 27, and

n_{13}=1mod 13. Only two possibilities: n_{13}=1 or 27.

If n_{13}=1, then the Sylow 13-subgroup is a normal subgroup of G, and we are done.

If n_{13}=27, then we are going to show that there can only be room for one Sylow 3-subgroup, and therefore the Sylow 3-subgroup is normal in G. We'll use the fact that distinct subgroups of order p for some prime p can only have the identity element in their intersection.

(Suppose P_1 and P_1 are subgroups of order p. Then $P_1 \cap P_2 \leq P_1$ and $P_1 \cap P_2 \leq P_1$. So $|P_1 \cap P_2|$ must be either 1 or p, and the only way it can be p is if $P_1 \cap P_2 = P_1$ and $P_1 \cap P_2 = P_2$, making $P_1=P_2$).

Therefore if P_1 and P_2 are not the same subgroup, their intersection has order 1, so contains only the identity element. Do be warned, though, that this is only true about subgroups of prime order, so this argument wouldn't work if, say, the Sylow 13-subgroups had order 13^2.

Since the Sylow 13-subgroups are subgroups of order 13, they can only intersect each other at the identity element. Also, every element of order 13 forms a subgroup of order 13, which has to be one of the Sylow 13-subgroups. Each Sylow 13 subgroup contains 12 elements of order 13 (every element except for the identity). There are 27 Sylow 13 sub-groups, so there are a total of 27 × 12=324 elements of order 13 in G.

This leaves 351-324=27 elements of G that do not have order 13. Since a Sylow 3-subgroup would have to have exactly 27 elements in it, this means that all these 27 elements form a Sylow 3-subgroup, and it must be the only one (since there are not any extra elements of G to use). So n_3=1 and this Sylow 3-subgroup must be normal in G.

Proposition 2.7 [5]

A group G is a direct product of subgroups H_1,H_2 if and only if

a) $G=H_1H_2$

b) $H_1 \cap H_2=\{e\}$ and

c) Every element of H_1 commutes with every element of H_2

Proof

If G is the direct product of H_1 *and* H_2, then certainly (a) and (c) hold and (b) holds because, for any $g \in H_1 \cap H_2$, the element (g, g^{-1}) maps to e under $(h_1,h_2) \rightarrow h_1h_2$ and so equals to (e,e).

Conversely, (c) implies that $(h_1,h_2) \rightarrow h_1h_2$ is a homomorphism and (b) implies that it is injective:

$h_1h_2=e \rightarrow h_1=h_2^{-1} \in H_1 \cap H_2=\{e\}$. Finally, (a) implies that it is surjective.

Proposition 2.8 [7,8]

A group G is a direct product of subgroups H_1,H_2 if and only if

a) $G=H_1,H_2$

b) $H_1 \cap H_2=\{e\}$

c) H_1 and H_2 are both normal in G

Proof

Certainly, these conditions are implied by those in proposition 1.37 and so it remains to show that they imply that each element h_1 of H_1 commutes with each element h_2 of H_2. Two elements h_1 and h_2 of G commute if and only if their commutator $[h_1,h_2] \stackrel{def}{=} (h_1h_2)(h_2h_1)^{-1}$, i.e., e but

$$\left(h_1h_2\right)\left(h_2h_1\right)^{-1} = h_1h_2h_1^{-1}h_2^{-1} = \begin{cases} (h_1h_2h_1^{-1}).\,h_2^{-1} \\ h_1.\left(h_2h_1^{-1}h_2^{-1}\right) \end{cases}$$

Which is in H_2 because H_2 is normal and is in H_1 because H_1 is normal. Therefore (b) implies that $[h_1,h_2]=e$.

Proposition 2.9 [9]

(i) Subgroups $H \subset G$ and quotient groups G/K of a solvable group G are solvable.

If normal subgroup $N \triangleleft G$ is solvable and if the quotient G/N is solvable then G is solvable.

Proof

$H(n) \subset G(n)$ for all n. Also $G(n)$ maps onto $(G/N)_{(n)}$ for all n.

If $(G/N)_{(n)} =\{1\}$ then $G(n) \subset N$. So if $N(m)=\{1\}$ then $G_{(N+M)} \subset N_{(M)}=\{1\}$

Corollary 2.10 [8]

All p-groups are solvable.

Proof

Let P be a p-group. We will induct on n, where $|P|=p^n$.

If $|P|$ = p, then P is cyclic. Therefore, P is abelian and, hence, solvable. Suppose that for all $1 \leq k < n$, groups of order P^k are solvable.

Let $|P|=p^n$ By Corollary 1, $Z.\ (P) \neq 1$ Thus, $|P/Z(P)|=p^k$ (for some k<n). So, by the inductive hypothesis, P=Z(P) is solvable. Since Z P is abelian, then Z(P) is solvable. Thus, by Proposition 1, P is solvable. Therefore, by induction, all p-groups are solvable.

Theorem 2.11 [9]

A finite group is nilpotent if and only if it is a direct product of its Sylow subgroups.

Proof

Finite p-groups are nilpotent by theorem 1.4 and hence so are any direct product of p-groups.

Now suppose that G is a finite nilpotent group and let P be any Sylow subgroup of G. Let $N=N_G(P)$. Suppose $N<G$. Then by Corollary 1 $N<N_G(N)$.

Let $x \in N_G(N)-N$ Hence $x^{-1}Px \leq x^{-1}N_X=N$ and so $x^{-1}Px$ is a Sylow subgroup of N as is P. It follows that $y^{-1}x^{-1}Pxy=P$ for some $y \in N$ and so $xy \in N$.

But $y \in N$ *and* $x \notin N$ a contradiction. It must therefore be that $N=G$ and so P is normal in G. G is therefore the direct product of its Sylow subgroups.

Theorem 2.12 [3]

Finite *p*-groups are nilpotent.

Proof

$Z_{r+1}(G)/Z_r(G)=Z(G/Z_r(G))$. Since the centre of a non-trivial *p*-group is non-trivial $Z_r(G)<Z_{r+1}(G)$ unless $Z_r(G)=G$.

Proposition 2.13 [3]

Every nilpotent group is solvable.

Proof

If G is nilpotent group then the upper central series of G.

$\{e\}=Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G) \subseteq \ldots Z_n(G)=G$ is a normal series.

Moreover for every i we have,

$Z_i(G)/Z_{i-1}=Z(G/Z_{i-1}(G))$

So all quotients of the upper central series are abelian.

Theorem 2.14

A normal subgroup N of a group G is a maximal normal subgroup if and only if the quotient G/N is simple.

RESULTS

Theorem 2.15

Let G be a group of order pq^r where $p=r$ and p,q,r distinct primes. The maximal subgroups of G are of order q^r and pq^{r-1} which are solvable.

Proof

Suppose G is a group of order pq^r. By theorem 2.1, the possibility for the order of subgroups of G are $p, pq, pq^2, \ldots pq^{r-1}, pq^r, q, q^2, \ldots, q^{r-1}, q^r$. It is clear that pq^{r-1} and q^r are the maximal subgroups of G since they are not contained in any other proper subgroups of G.

It remains to show that the two maximal subgroups are solvable and nilpotent. q^r is solvable by Corollary 2.10.

What left is to show that pq^{r-1} is solvable. Let G_1 be a group of order pq^{r-1}, the sylow p and q subgroups of G_1 are normal in the group. Let H_1 be the Sylow p-subgroup of G_1 then, $|G_1/H_1| = q^{r-1}$. Since q is prime then G_1/H_1 and H_1 are solvable by corollary 2.10, it follows that G_1 is solvable by (ii) of proposition 2.

APPLICATION

Consider the permutation groups C_1 and D_1

$C_1=\{(1),(123),(132)\}, D_1=\{(1),(46)\}$ acting on the sets $S_1=\{123\}$ *and* $\Delta_1=\{46\}$, respectively.

Let $P = C_1^{(\Delta_1)} = f : \Delta_1 \to C_1 \textit{then} \ |P| = |C_1|^{(\Delta_1)} = 3^2 = 9$

We can easily verify that G_1 is a group with respect to the operations

$(f_1 f_2)\delta_1 = f_1(\delta_1) f_2(\delta_1)$ *where* $\delta_1 \epsilon \Delta_1$.

The wreath products of C_1 and D_1 is given by W_1 as it appears in the result validation which is given by the (GAP) below:

gap>C1:=Group ((1, 2, 3));

Group ([(1, 2, 3)])

gap>D1:=Group ((4, 6));

Group ([(4, 6)])

gap>W1:=Wreath Product (C1, D1);

Group ([(1, 2, 3), (4, 5, 6), (1, 4) (2, 5) (3, 6)])

gap>Maximal Subgroup Class Reps (W1);

[Group ([(1, 3, 2) (4, 6, 5) (1, 2, 3) (4, 6, 5)]), Group ([(1, 4) (2, 5) (3, 6), (1, 2, 3) (4, 6, 5)]), Group ([(1, 4) (2, 5) (3, 6), (1, 3, 2) (4, 6, 5)])]

gap>M1:=Group ([(1, 3, 2) (4, 6, 5) (1, 2, 3) (4, 6, 5)]);

Group([(1, 3, 2) (4, 6, 5) (1, 2, 3) (4, 6, 5)])

gap>Is Solvable (M1);

true

gap>Is Nilpotent (M1);

true

gap>Is Normal (W1, M1);

true

gap>Is Simple (M1);

false

gap>Is Transitive (M1);

false

gap>Is Primitive (M1);

false

gap>M2:=Group ([(1, 4) (2, 5) (3, 6) (1, 2, 3) (4, 6, 5)]);

Group ([(1, 4) (2, 5) (3, 6) (1, 2, 3) (4, 6, 5)])

gap>Is Solvable (M2);

true

gap>Is Nilpotent (M2);

false

gap>Is Normal (W1, M2);

true

gap>Is Simple (M2);

false

gap>Is Transitive (M2);

true

gap>Is Primitive (M2);

false

gap>M3:=Group ([(1, 4) (2, 5) (3, 6) (1, 3, 2) (4, 6, 5)]);

Group ([(1, 4) (2, 5) (3, 6) (1, 3, 2) (4, 6, 5)])

gap>Is Solvable (M3);

true

gap>Is Nilpotent (M3);

true

gap>Is Normal (W1, M3);

false

gap>Is Simple (M3);

false

gap>Is Transitive (M3);

true

gap>Is Primitive (M3);

false

gap>quit;

Consider the permutation groups C_2 and D_2

C_2={(1),(15432),(14253),(13524), (12345)}

D_2={(1),(678),(687)} acting on the sets S_2={1,2,3,4,5} and Δ_2={6,7,8}, respectively.

Let $P = C_2^{\Delta_2} = \{f : \Delta_2 \to C_2\}$ *then* $|P| = |C_2|^{\Delta_2} = 3^4 = 81$

We can easily verify that G_1 is a group with respect to the operations

$(f_1 f_2)\delta_1 = f_1(\delta_1) f_2(\delta_1)$ *where* $\delta_1 \epsilon \Delta_1$

The wreath products of C_2 and D_2 is given by W_1 as it appears in the result validation which is given by the (GAP) below:

gap>C2:=Group ((1, 2, 3, 4, 5));

Group ([(1, 2, 3, 4, 5)])

gap>D2:=Group ((6, 7, 8));

Group ([(6, 7, 8)])

gap>W2:=Wreath Product (C2, D2);

Group ([(1, 2, 3, 4, 5) (6, 7, 8, 9, 10) (11, 12, 13, 14, 15) (1, 6, 11) (2, 7, 12) (3, 8, 13) (4, 9, 14) (5, 10, 15)])

gap>Maximal Subgroup Class Reps (W2);

[Group ([(1, 3, 5, 2, 4) (6, 8, 10, 7, 9) (11, 13, 15, 12, 14) (1, 2, 3, 4, 5) (11, 15, 14, 13, 12) (1, 5, 4, 3, 2) (6, 7, 8, 9, 10)])

Group ([(1, 6, 11) (2, 7, 12) (3, 8, 13) (4, 9, 14) (5, 10, 15) (1, 2, 3, 4, 5) (11, 15, 14, 13, 12) (1, 5, 4, 3, 2) (6, 7, 8, 9, 10)])

Group ([(1, 6, 11) (2, 7, 12) (3, 8, 13) (4, 9, 14) (5, 10, 15) (1, 3, 5, 2, 4) (6, 8, 10, 7, 9) (11, 13, 15, 12, 14)])]

gap>M4:=Group ([(1, 3, 5, 2, 4) (6, 8, 10, 7, 9) (11, 13, 15, 12, 14) (1, 2, 3, 4, 5) (11, 15, 14, 13, 12) (1, 5, 4, 3, 2) (6, 7, 8, 9, 10)]);

Group ([(1, 3, 5, 2, 4) (6, 8, 10, 7, 9) (11, 13, 15, 12, 14) (1, 2, 3, 4, 5) (11, 15, 14, 13, 12) (1, 5, 4, 3, 2) (6, 7, 8, 9, 10)])

gap>Is Solvable (M4);

true

gap>Is Nilpotent (M4);

true

gap>Is Normal (W2, M4);

true

gap>Is Simple (M4);

false

gap>Is Transitive (M4);

false

gap>Is Primitive (M4);

false

gap>M5:=Group ([(1, 6, 11) (2, 7, 12) (3, 8, 13) (4, 9, 14) (5, 10, 15) (1, 2, 3, 4, 5) (11, 15, 14, 13, 12) (1, 5, 4, 3, 2) (6, 7, 8, 9, 10)]);

Group ([(1, 6, 11) (2, 7, 12) (3, 8, 13) (4, 9, 14) (5, 10, 15) (1, 2, 3, 4, 5) (11, 15, 14, 13, 12) (1, 5, 4, 3, 2) (6, 7, 8, 9, 10)])

gap>Is Solvable (M5);

true

gap>Is Nilpotent (M5);

false

gap>Is Normal (W2, M5);

true

gap>Is Simple (M5);

false

gap>Is Transitive (M5);

true

gap>Is Primitive (M5);

false

gap>M6:=Group ([(1, 6, 11) (2, 7, 12) (3, 8, 13) (4, 9, 14) (5, 10, 15) (1, 3, 5, 2, 4) (6, 8, 10, 7, 9) (11, 13, 15, 12, 14)]);

Group ([(1, 6, 11) (2, 7, 12) (3, 8, 13) (4, 9, 14) (5, 10, 15) (1, 3, 5, 2, 4) (6, 8, 10, 7, 9) (11, 13, 15, 12, 14)])

gap>Is Solvable (M6);

true

gap>Is Nilpotent (M6);

false

gap>Is Normal (W2, M6);

true

gap>Is Simple (M6);

false

gap>Is Transitive (M6);

true

gap>Is Primitive (M6);

false

gap>quit;

Consider the permutation groups C_3 and D_3

C_3={(1),(15432),(14253),(13524),(12345)}

D_3={(1),(678),(687)} acting on the sets S_3={1,2,3,4,5} and Δ_3={6,7,8}, respectively.

Let $P = C_2^{\Delta_2} = \{ f : \Delta_2 \to C_2 \}\, then |P| = |C_2|^{\Delta_2} = 3^4 = 81$

We can easily verify that G_1 is a group with respect to the operations

$(f_1f_2)\delta_1 = f_1(\delta_1)f_2(\delta_1)$ where $\delta_1 \epsilon\, \Delta_{1.}$

The wreath products of C_3 and D_3 is given by W_3 as it appears in the result validation which is given by the (GAP) below:

gap>C3:=Group ((1, 2, 3));

Group([(1, 2, 3)])

gap>D3:=Group ((4, 5, 6, 7, 8));

Group ([(4, 5, 6, 7, 8)])

gap>W3:=Wreath Product (C3, D3);

Group ([(1, 2, 3) (4, 5, 6) (7, 8, 9) (10, 11, 12) (13, 14, 15) (1, 4, 7, 10, 13) (2, 5, 8, 11, 14) (3, 6, 9, 12, 15)])

gap>Maximal Subgroup Class Reps (W3);

[Group ([(1, 4, 7, 10, 13) (2, 5, 8, 11, 14) (3, 6, 9, 12, 15) (1, 2, 3) (13, 15, 14) (1, 2, 3) (10, 12, 11) (1, 2, 3) (7, 9, 8)

(1, 3, 2) (4, 5, 6)])

Group ([(1, 3, 2) (4, 6, 5) (7, 9, 8) (10, 12, 11) (13, 15, 14) (1, 2, 3) (13, 15, 14) (1, 2, 3) (10, 12, 11) (1, 2, 3) (7, 9, 8) (1, 3, 2) (4, 5, 6)])

Group ([(1, 4, 7, 10, 13) (2, 5, 8, 11, 14) (3, 6, 9, 12, 15) (1, 3, 2) (4, 6, 5) (7, 9, 8) (10, 12, 11) (13, 15, 14)])]

gap>M7:=Group ([(1, 4, 7, 10, 13) (2, 5, 8, 11, 14) (3, 6, 9, 12, 15) (1, 2, 3) (13, 15, 14) (1, 2, 3) (10, 12, 11) (1, 2, 3) (7, 9, 8) (1, 3, 2) (4, 5, 6)]);

Group ([(1, 4, 7, 10, 13) (2, 5, 8, 11, 14) (3, 6, 9, 12, 15) (1, 2, 3) (13, 15, 14) (1, 2, 3) (10, 12, 11) (1, 2, 3) (7, 9, 8) (1, 3, 2) (4, 5, 6)])

gap>Is Solvable (M7);

true

gap>Is Nilpotent (M7);

false

gap>Is Normal (W3, M7);

true

gap>Is Simple (M7);

false

gap>Is Transitive (M7);

true

gap>Is Primitive (M7);

false

gap>M8:=Group ([(1, 3, 2) (4, 6, 5) (7, 9, 8) (10, 12, 11) (13, 15, 14) (1, 2, 3) (13, 15, 14) (1, 2, 3) (10, 12, 11) (1, 2, 3) (7, 9, 8) (1, 3, 2) (4, 5, 6)]);

Group ([(1, 3, 2) (4, 6, 5) (7, 9, 8) (10, 12, 11) (13, 15, 14) (1, 2, 3) (13, 15, 14) (1, 2, 3) (10, 12, 11) (1, 2, 3) (7, 9, 8) (1, 3, 2) (4, 5, 6)])

gap>Is Solvable (M8);

true

gap>Is Nilpotent (M8);

true

gap>Is Normal (W3, M8);

true

gap>Is Simple (M8);

false

gap>Is Transitive (M8);

false

gap>Is Primitive (M8);

false

gap>Maximal Subgroup Class Reps (W3);

[Group ([(1, 4, 7, 10, 13) (2, 5, 8, 11, 14) (3, 6, 9, 12, 15) (1, 2, 3) (13, 15, 14) (1, 2, 3) (10, 12, 11) (1, 2, 3) (7, 9, 8) (1, 3, 2) (4, 5, 6)])

Group ([(1, 3, 2) (4, 6, 5) (7, 9, 8) (10, 12, 11) (13, 15, 14) (1, 2, 3) (13, 15, 14) (1, 2, 3) (10, 12, 11) (1, 2, 3) (7, 9, 8) (1, 3, 2) (4, 5, 6)])

Group ([(1, 4, 7, 10, 13) (2, 5, 8, 11, 14) (3, 6, 9, 12, 15) (1, 3, 2) (4, 6, 5) (7, 9, 8) (10, 12, 11) (13, 15, 14)])]

gap>M9:=Group ([(1, 4, 7, 10, 13) (2, 5, 8, 11, 14) (3, 6, 9, 12, 15) (1, 3, 2) (4, 6, 5) (7, 9, 8) (10, 12, 11) (13, 15, 14)]);

Group ([(1, 4, 7, 10, 13) (2, 5, 8, 11, 14) (3, 6, 9, 12, 15) (1, 3, 2) (4, 6, 5) (7, 9, 8) (10, 12, 11) (13, 15, 14)])

gap>Is Solvable (M9);

true

gap>Is Nilpotent (M9);

true

gap>Is Normal (W3, M9);

false

gap>Is Simple (M9);

false

gap>Is Transitive (M9);

true

gap>Is Primitive (M9);

false

gap>

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