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ON THE BOUNDEDNESS AND STABILITY OF SOLUTIONS OF CERTAIN THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

The objective of this paper is to give sufficient conditions for the existence of bounded solutions that are globally exponentially stable, periodic and almost periodic for a certain third-order non-linear differential equation. A matrix inequality is obtained and proved to satisfy a generalized frequency domain inequality of [7] through the frequency domain technique.

Keywords: Boundedness, globally exponentially stable, periodic or almost periodic, Lyapunov function.

INTRODUCTION

In a relatively recent note of Clen Jipeng (1998) [9], a suitable Lyapunov function of the type "quadratic form only" was used to obtain sufficient conditions for the existence of a solution which is uniformly ultimately bounded and periodic (or almost periodic) for the third- order non-linear differential equation.

$$x''' + F(r)\,x'' + G(r)\,x' + H(r)\,x = e(x) \qquad (1.1)$$
$$G(r) = 1 - \varepsilon[1 + km(\varepsilon)r^2],$$
$$H(r) = 1 - \varepsilon kl(\varepsilon)\,r^2.$$

The functions $F(r), G(r), H(r), and\ e(x)$ are continuous with $r^2 = x^2 + x'^2 + x''^2$. Moreover $K > 0\ and\ \varepsilon$ are real parameters. Equations of the form (1.1) with combinations of non linear terms have been of great interest to many mathematicians for decades. The reader can find interesting expositions in [2, 3, 4, 5, 8, 12, 13]. These equations are not only of theoretical interest, but also of a great practical importance as they can be applied to model automatic control in T.V systems realized by means of R-C filters.

In this study, I shall also consider the equation (1.1) and find necessary and sufficient conditions that guarantees the existence of a bounded solution, which is globally exponentially stable and periodic (or almost periodic). The method of approach is the frequency domain technique. The frequency domain technique employed in this study is of beneficial applications and circumvents the limitations experienced in the practical construction of the well known Lyapunov functions method. Besides, it has been asserted [10] that no Lyapunov function can be better than the frequency domain inequality criteria. This fact is more profound in the work of Barbalat and Halancy [7] and Yacubovish [15, 16]. For more exposition on the frequency domain method, see [1, 2, 11]. My ~~ine~~ approach in this study has an advantage over the Lyapunov second method in [9], because the best choice of Lyapunov function of the type "quadratic form plus the integral of the non linear term" was not used in [9]. Consequently, the results obtained ~~in~~ by [9] cannot be better than the result obtained in this study.

In an interesting paper, Afuwape [1] derived conditions for the existence of solutions that are bounded, globally exponentially stable and periodic (or almost periodic) for special cases of the equation (1.1), when the non linear terms $F, G, H \; and \; e(x)$ depend only on one argument. The results obtained in this study improved some of those contained in [2] and [9]. This study depends on the generalized Yacubovish's theorem [7], which is stated without proof.

Generalised Yacubovish's Theorem [7].

Consider the system:

$$x' = AX - B\varphi(G) + D(t), \sigma = C^x X$$

(1.2)

Where A is an n× n real matrix, B and C are n× m real matrices with C^x as the transpose of $C, \varphi(\sigma) = Col\; \varphi$j $(\sigma$j$)$, (j $= 1, 2, - - - -, m)$ $and\; D(t)$ is an n-vector.

Suppose that in the system (1.3), the following assumptions are true:

(i) A is a stable matrix;

(ii) $D(t)$ is bounded for all $t \; in \; R$;

(iii) For some constants $\hat{U}j \geq 0$, (j $= 1, 2, - - - - -, m)$.

(iv) There exist a diagonal matrix D ≥ 0, such that the frequency domain inequality,

$$\pi\,(\omega) = MD + Re\, DG(i\omega) > 0$$

(1.4)

holds for all $\omega \; in \; R$, where $G(i\omega) = Cx\,(i\omega I - A) - 1B$ is the transfer function and $M = diag(1/\,\hat{U}j)$, (j $= 1, 2, - - - - -, m)$.

Then, the system (1.2) has a unique solution which is bounded in R, globally exponentially stable and periodic (or almost periodic) whenever $D(t)$ is periodic (or almost periodic).

This paper is organized in the following ways. In section 2, preliminary notes which will be needed in the next sections was given. In section 3, the main results of the work and part of the proof will be given. The last section will conclude the proof of the main result.

2.0 Preliminary Notes

The equation (1.1) can be transformed into its equivalent non-linear system by setting,

$$x' = y,$$
$$y' = \cancel{z,}$$

$z' = -x - (1-\varepsilon)y - (1-\varepsilon)z + k\,\varepsilon\,[l(\varepsilon)x + m(\varepsilon)y + n(\varepsilon)z]\,(x^2 + y^2 + z^2) + e(t)$ (2.1)

This system is a periodic system with period, ω and satisfies the uniqueness condition of the solution with respect to the initial value problem on product space 1×F, where $t\,E \in I, I = (0 + \infty), [x, y, z] \in F. F$ is an arbitrary compact subset of R^3.

The linear part of the system (2.1) is the system,

$x' = y,$

$y' = z,$

$$z' = -x - (1-\varepsilon)y - (1-\varepsilon)z, \qquad (2.2)$$

from which we can derive coefficient matrix A as,

$$A(\mathcal{E}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -(1-\mathcal{E}) & -(1-\mathcal{E}) \end{pmatrix} \qquad (2.3)$$

With characteristics polynomials,

$$det(\lambda I - A) = \lambda 3 + (1-\mathcal{E})\,\lambda 2 + (1-\mathcal{E})\,\lambda + 1 \qquad (2.4)$$

The characteristics roots of the equation (2.4) are given as,

$$\lambda_1 = -1$$

$$\lambda_2 = \frac{\mathcal{E}}{2} + \left(1 - [\tfrac{\mathcal{E}}{2}]^2\right)^{\frac{1}{2}} \qquad (2.5)$$

$$\lambda_3 = \frac{\mathcal{E}}{2} + \left(1 - [\tfrac{\mathcal{E}}{2}]^2\right)^{\frac{1}{2}}$$

NOTE that for matrix to be stable, all it's eigenvalues should have negative real parts, i.e. $Re(\lambda j) < 0$. For our case, $j = 1, 2, 3$, and moreover A(Ɛ) will be stable if $\mathcal{E} \leq -2$. Thus we can have $(i\omega I - A(\mathcal{E}))$ to be,

$$(i\omega I - A(\mathcal{E})) = \begin{pmatrix} i\omega & -1 & 0 \\ 0 & i\omega & -1 \\ 1 & (1-\mathcal{E}) & i\omega+(1-\mathcal{E}) \end{pmatrix} \qquad (2.6)$$

And,

$$det((i\omega I - A(\mathcal{E})) = \Delta = \omega 2(1-\mathcal{E}) + 1 - i\omega(\omega 2 + \mathcal{E} - 1) \qquad (2.7)$$

From which we get,

$$|\Delta|2 = [\,\omega 2(\mathcal{E}-1) + 1]^2 + \omega 2\,[\,\omega 2 + \mathcal{E} - 1]^2.$$

RESULTS

Theorem 3.1: Suppose that in the equation (1.1), there exist positive parameters μ_1, μ_2 and μ_3 such that for all $Z, \widehat{Z} \in R,\ Z \neq \hat{Z}$, we have

$$0 \leq \frac{H(Z)-H(\widehat{Z})}{Z-\widehat{Z}} \leq \mu_1 \quad (3.1)$$

$$0 \leq \frac{G(Z)-G(\hat{Z})}{Z-\hat{Z}} \leq \mu_2 \quad (3.2)$$

$$0 \leq \frac{F(Z)-F(\widehat{Z})}{Z-\hat{Z}} \leq \mu_3 \quad (3.3)$$

And the inequality,

$$\mu_1 - \tau_1\mu_1\left(\frac{\mu 3}{\tau 3} + \frac{\mu 2}{\tau 2}\right) > б(\mathcal{E} - 1)\,\mu_1\mu_2\mu_3, \quad (3.4)$$

is satisfied. Then the equation (1.1) has a solution which is bounded in ~~$R1$~~ R_1 globally exponentially stable and periodic or almost periodic according as $e(t)$ is periodic or almost periodic.

The proof of the main result. Let us set $x' = y$ in the equation (1.1) to have the system (2.1). (1.2) can be rewritten in the vector form,

$$x' = AX - B\varphi(б) + D(t), \text{ where } б = C^x X$$

Also, $A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -(1\text{-}\mathcal{E}) & -(1\text{-}\mathcal{E}) \end{pmatrix}$; $B=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$; $C=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $P(t)\begin{pmatrix} 0 \\ 0 \\ e(t) \end{pmatrix}$;

$$B\varphi(б)\begin{pmatrix} \hat{H}(x) \\ \hat{G}(y) \\ \mathrm{F}(z) \end{pmatrix} \quad (3.5)$$

The transfer function $G(i\omega) = c^x(i\omega I - A)^{-1}B$ of this system thus becomes,

$$G(i\omega) = 1/\Delta \begin{pmatrix} 1 & 1 & 1 \\ i\omega & i\omega & i\omega \\ -\omega 2 & -\omega 2 & -\omega 2 \end{pmatrix} \quad (3.6)$$

On choosing,

$$D=\begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 1 & 1 & \tau_3 \end{pmatrix}, \quad \text{and } M = \begin{pmatrix} \frac{1}{\mu_1} & 0 & 0 \\ 0 & \frac{1}{\mu_2} & 0 \\ 0 & 0 & \frac{1}{\mu_3} \end{pmatrix}$$

We obtain the frequency domain inequality of (1.4) as,

$$\pi(\omega)=\begin{pmatrix}\pi_{11} & \pi_{12} & \pi_{13}\\ \pi_{21} & \pi_{22} & \pi_{23}\\ \pi_{31} & \pi_{32} & \pi_{33}\end{pmatrix}>0 \qquad (3.7)$$

where,

$$\pi_{11}(\omega)=\tau_1[\frac{1}{\mu_1}+\frac{\omega^2(\mathcal{E}-1)+1}{|\Delta|^2}], \qquad (3.8)$$

$$\pi_{22}(\omega)=\tau_2[\frac{1}{\mu_2}-\frac{\omega^2(\omega^2+\mathcal{E}-1)}{|\Delta|^2}], \qquad (3.9)$$

$$\pi_{33}(\omega)=\tau_3[\frac{1}{\mu_3}-\frac{\omega^2(\omega^2(\mathcal{E}-1)+1}{|\Delta|^2}], \qquad (3.10)$$

$$\pi_{12}(\omega)=\frac{\tau_{1\bar{\Delta}}}{|\Delta|^2}-i\omega\tau_2\Delta=\bar{\pi}_{21}(\omega), \qquad (3.11)$$

$$\pi_{13}(\omega)=(\frac{\bar{\Delta}}{2|\Delta|^2}-\omega^2\tau_3\Delta)=\bar{\pi}_{13}, \qquad (3.12)$$

$$\pi_{23}(\omega)=(\frac{i\omega\tau_2\bar{\Delta}}{2|\Delta|^2}-\omega^2\tau_3\Delta)=\bar{\pi}_{32}, \qquad (3.13)$$

For us to show that the inequality (3.7) is valid, it is suffices to use the Sylvester criteria which states that the principal minors of $\pi(\omega)$ in (3.7) be strictly positive. We shall now prove these in series of Lemma.

Lemma 1: For all $\omega \; in \; R,\; \pi ii\;(\omega) > 0, (i = 1, 2, 3)$.

PROOF: For all $i = 1, \pi ii\;(\omega)$ in the equation (3.8) will be positive for all $\omega \; in \; R$.

If $$\mu 1 < v(\mathcal{E}-1)+1-\frac{v(v+\mathcal{E}-1)^2}{v(\mathcal{E}-1)+1} \quad ,\omega^2\equiv v \qquad (3.14)$$

Let,

$$s_1(v) = -v(\mathcal{E}-1)+1-\frac{v(v+\mathcal{E}-1)}{v(\mathcal{E}-1)+1},$$

Then $v_1 = 0$, $v_2 = 1-\mathcal{E} \; and \; v_3 = -1/\mathcal{E}-1$,

Are the roots of the equations $v(v+\mathcal{E}-1)^2 = 0 \; and \; (v(\mathcal{E}-1)+1)^2 = 0$ respectively. We shall denote by m_1, the minimum of $s_1(v)$ and let this minimum be attained at say, $v = v_0$. Then $s_1^1(v_0) = 0$. Thus, $s_1^1(v_0)$ can be zero in the interval $[v_1, v_2]$, obviously the points v_1 and v_2 are the minimum points of $s_1(v)$. If we substitute $v = v_1$ and $v = v_2$ respectively in the inequality (3.14), we shall have $s_1(v_1)= 1$ and $s_2(v_2)= \mathcal{E}^2-2\mathcal{E}+2$. We note that there is an asymptote at $v = v_3$ for $s_1(v)$.

Thus the minimum value of $s_1(v) = m_1$ and it is attainable in the interval $[v_1, v_2]$. hence $\pi_{11}(\omega) > 0$. Next, for $i = 2$, $\pi_{22}(\omega)$ in the equation (3.9) will be positive for all $\omega \; in \; R$, if we can show that,

$$\mu_2 < v(v+\mathcal{E}-1)+\frac{(v(\mathcal{E}-1)+1)^2}{v(v+\mathcal{E}-1)}. \qquad (3.15)$$

Let,

$$s_2(v) = \mu_2 < v(v+\mathcal{E}-1)+\frac{v(\mathcal{E}-1)+1)^2}{v(v+\mathcal{E}-1)}.$$

We note that there are asymptotes at $s_2(v_1)$ and $s_2(v_2)$. The maximum that $s_2(v)$ can attain is at say $v = v_0$. Let $m_2 = s_2(v_0)$ be this maximum which is attainable at $v = v_3$, and given by,

$$s_2(v_3) = \frac{\mathcal{E}(1-(\mathcal{E}-1))}{(\mathcal{E}-1)^2}$$

(3.16)

Thus $\pi_{22}(\omega) > 0$. Next, for $i = 2$, $\pi_{33}(\omega)$ in the equation (3.10) will be positive for all ω in R, if we can show that,

$$\mu_3 < \frac{v(\mathcal{E}-1)+1}{v} + \frac{(v+\mathcal{E}-1)}{v(\mathcal{E}-1)+1}$$

(3.17)

This is possible if we let,

$$s_3(v) = \frac{v(\mathcal{E}-1)+1}{v} + \frac{(v+\mathcal{E}-1)}{v(\mathcal{E}-1)+1},$$

and show that it has a maximum which is always negative. Let $v = v_0$, be the point at which this maximum (m_3) is attained. Observe that at $s_3(v_1)$ and $s_3(v_3)$, there are asymptotes, hence the maximum is not attainable at these points. At the point $v = v_2$,

$$s_3(v_2) = \frac{\mathcal{E}(2-\mathcal{E})}{1-\mathcal{E}},$$

(3.18)

And the maximum is attainable there, thus $\pi_{33}(\omega) > 0$.

LEMMA 2: For all $\omega\ in\ R$,

$\pi_{ii}(\omega)\pi_{jj}(\omega) - |\pi_{ij}(\omega)|^2 > 0, \quad (i \neq j;\ i, j, = 1, 2, 3)$

Proof: for $i = 1, j = 2$, we derive from the inequality (3.7),

$$\pi_c(\omega) = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} > 0. \qquad (3.19)$$

For us to show that the inequality (3.19) is satisfied, it suffices to use Sylvester`s criteria which states that the principal minors of the matrix in (3.19) and det $\pi_c(\omega)$ be positive definite. It has already been shown in the proof of the Lemma 1 that both $\pi_{11}(\omega)$ and $\pi_{22}(\omega)$ are positive. It remains to show that det $\pi_c(\omega)$ in the inequality (3.19) is positive definite.

On simplifying the inequality (3.19), we have,

$$\pi_c(\omega) = \pi_{11}\pi_{22} - |\pi_{12}|^2 = \frac{\tau_1\tau_2}{\mu_1\mu_2} - \frac{\omega^2\tau_1\tau_2(\omega^2+\mathcal{E}-1)}{\mu_1|\Delta|^2} + \frac{\tau_1\tau_2(\omega^2+(\mathcal{E}-1)+1}{\mu_2|\Delta|^2} - \frac{\tau_1^2+\omega^2\tau_2^2}{4|\Delta|^2} > 0 \qquad (3.20)$$

Further simplification give,

$$\pi_c(\omega) = \frac{\tau_1\tau_2}{\mu_1\mu_2}(\omega^6 + \omega^4)[(\mathcal{E}-1)^2 + 2(\mathcal{E}-1) - \mu_2] + \omega^2[(\mathcal{E}-1)^2 + (2 + \mu_1 - \mu_2)(\mathcal{E}\text{-}1) - \frac{\tau_2\mu_1\mu_2}{4\tau_1} + (\mu_1 + 1 - \frac{\tau_2\mu_1\mu_2}{4\tau_1}] > 0.$$

This will be true if,

$$(\mu_1 - \mu_2)(\mathcal{E}-1) - \frac{\tau_2}{4\tau_1}\mu_1\mu_2, \text{ and } \frac{4(1+\mu_1)}{\mu_1\mu_2} > \frac{\tau_1}{\tau_2},$$

Hence we have,

$$\frac{\mu_1\mu_2}{4(1+\mu_1)} < \frac{\tau_2}{\tau_1} < \frac{4(\mu_1-\mu_2)}{\mu_1\mu_2},$$

Which imply that,

$$(\mu_1\mu_2)^2 < 16(\mu_1 - \mu_2)(1+\mu_1).$$

Thus,

$$\pi_c(\omega) = 0.$$

Next is to show that,

$$\pi_d(\omega) = \begin{pmatrix} \pi_{11} & \pi_{13} \\ \pi_{31} & \pi_{33} \end{pmatrix} > 0. \qquad (3.21)$$

On following the above arguments, we only need to show that det $\pi_d(\omega)$ in (3.21) is positive definite so as to satisfy Sylvester`s criteria, since it has already been proved in the Lemma 1 that π_{11} and π_{13} are respectively positive.
On simplifying the inequality (3.21), we obtain,

$$\pi_d(\omega) = \pi_{11}\pi_{33} - |\pi_{13}|^2 = \frac{\tau_1\tau_3}{\mu_1\mu_3} - \frac{\omega^2\tau_1\tau_3(\omega^2(\mathcal{E}-1)+1}{\mu_1|\Delta|^2} + \frac{\tau_1\tau_3(\omega^2(\mathcal{E}-1)+1}{\mu_3|\Delta|^2} - \frac{\tau_1^2+\omega^4\tau_3^2}{4|\Delta|^2} + \frac{\omega^2\tau_1\tau_3(\omega^2(\mathcal{E}-1)+1)^2-2\omega^4\tau_1\tau_3(\omega^2+(\mathcal{E}-1)}{|\Delta|^4} > 0,$$

From which we have,

$$4|\Delta|^2 + |\Delta|^2[(\omega^2(\mathcal{E}-1)+1)(\mu_1-\mu_3)\frac{1}{4}(\frac{\tau_1}{\tau_3} + \frac{\omega^2 4\tau_3}{\tau_1})]\,\mu_1\mu_3 > [2\omega^4(\omega^2+(\mathcal{E}-1))^2 - \omega^2\,(\omega^2(\mathcal{E}-1)+1)^2]\mu_1\mu_3. \qquad (3.22)$$

For the inequality (3.22) to be valid, it suffices to show that the minimum of its left hand side is positive and is greater than the maximum of the right hand side. The maximum of the right hand side is $-2\mu_1\mu_3$ and the minimum of the left hand side is $1+\mu_1-\frac{\tau_1\mu_1\mu_3}{4\tau_3}$, thus we have,

$$1+\mu_1-\frac{\tau_1\mu_1\mu_3}{4\tau_3} > -2\mu_1\mu_3,$$

Which implies that,

$$\frac{4+(1+\mu_1)}{\mu_1\mu_3} + 8 > \frac{\tau_1}{\tau_3}.$$

Hence, $\pi_d(\omega) > 0$.
At least we shall show that,

$$\pi_e(\omega) = \begin{pmatrix}\pi_{22} & \pi_{23}\\ \pi_{32} & \pi_{33}\end{pmatrix} > 0$$

(3.23)
On using the preceding arguments, we have,

$$\pi_e(\omega) = \pi_{22}\pi_{33} - |\pi_{23}|^2 = \frac{\tau_2\tau_3}{\mu_2\mu_3} - \frac{\omega^2\tau_2\tau_3(\omega^2(\mathcal{E}-1)+1}{\mu_2|\Delta|^2} - \frac{\omega^2\tau_2\tau_3(\omega^2+(\mathcal{E}-1))}{\mu_3|\Delta|^2} - \frac{\omega^2(\tau_2^2+\tau_3^2)}{4|\Delta|^2} > 0.$$

Which reduces to,

$$\omega^6 + \omega^4[(\mathcal{E}-1)^2 + (\mathcal{E}-1) - \mu_2 - \tau_3\mu_2\mu_3] + \omega^2[(\mathcal{E}-1)^2 + (2-\mu_2)(\mathcal{E}-1) - \mu_3 -] \frac{\tau_2\mu_2\mu_3}{4\tau_3}] + 1 > 0.$$

(3.24)
The inequality (3.24) holds if,

$$\frac{(\mathcal{E}-1)-\mu_2}{\mu_2\mu_3} > \frac{\tau_3}{4\tau_2}, \text{ and } \frac{(2-\mu_2)(\mathcal{E}-1)-\mu_3}{\mu_2\mu_3} > \frac{\tau_2}{4\tau_3}.$$

This is possible since,

$$\frac{\mu_2\mu_3}{4(\mathcal{E}-1)-\mu_3} < \frac{\tau_2}{\tau_3} < 4\left(\frac{(2-\mu_2)(\mathcal{E}-1)-\mu_3}{\mu_2\mu_3}\right),$$

From which we have,

$$(\mu_2\mu_3)^2 < 16[((2-\mu_2)(\mathcal{E}-1)-\mu_3)((\mathcal{E}-1)-\mu_2).$$

4. CONCLUSION TO THE PROOF OF THE MAIN RESULT.
It now remains to show that from the inequality (3.7),

$$\det\pi(\omega) = \pi_{11}(\pi_{22}\pi_{33} - |\pi_{23}|^2 + \pi_{22}\left(\pi_{11}\pi_{33} - |\pi_{31}|^2\right) + \pi_{23}\left(\pi_{11}\pi_{23} - |\pi_{12}|^2\right) - 2\pi_{11}\pi_{22}\pi_{33} + 2\mathrm{Re}(\pi_{12}\pi_{23}\pi_{31}) > 0. \quad (4.1)$$

This is equivalent on further simplification to,

$$\det\pi(\omega) = \frac{\tau_1\tau_2\tau_3}{\mu_1\mu_2\mu_3} - \frac{1}{|\Delta|^2}\left(\frac{\omega^2}{\mu_1\mu_2}\tau_1\tau_2\tau_3(\omega^2(\mathcal{E}-1)+1\right) + \frac{\omega^2\tau_1}{4\mu_1}(\tau_2^2+\omega^2\tau_3^2) + \frac{\tau_2}{4\mu_2}(\tau_1^2+\omega^4\tau_3^2) + \frac{\tau_1}{4\mu_3}(\tau_1^2+\omega^2\tau_2^2) - \frac{\tau_1\tau_2\tau_3}{\mu_2\mu_3}(\omega^2(\mathcal{E}-1)+1)^2 - \frac{\omega^2\tau_1\tau_2\tau_3}{\mu_2|\Delta|^4}(\omega^2(\mathcal{E}-1)+1)^2 - 2\omega^2(\omega^2+\mathcal{E}-1)^2 + \frac{\mu_2}{4\tau_1\tau_3}(\tau_1^2+\omega^4\tau_3^2)(\omega^2+\mathcal{E}-1) - \frac{\omega^4}{|\Delta|^6}(\omega^2+\mathcal{E}-1)(2\omega^2(\omega^2+\mathcal{E}-1)^2 - 3(\omega^2(\mathcal{E}-1)+1^2)\tau_1\tau_2\tau_3 > 0. \quad (4.2)$$

For $\omega^2 = 0$, $\det\pi(\omega)$ is positive if,

$$1-\mu_1 > \frac{\mu_1\tau_1}{4}\left[\frac{\mu_3}{\tau_3}+\frac{\mu_2}{\tau_2}\right].$$

For $\omega^2 \neq 0$, $\det\pi(\omega)$ will be positive if we can show that,

$$|\Delta|^6 - |\Delta|^4(\omega^2\mu_3[\omega^2(\mathcal{E}-1)+1] + \omega^2\frac{\mu_2\mu_3}{4\tau_2\tau_3}(\tau_2^2+\omega^2\tau_3^2) + \frac{\mu_1\mu_3}{4\tau_1\tau_3}(\tau_1^2+\omega^4\tau_3^2) + \frac{\mu_1\mu_2}{4\tau_1\tau_2}(\tau_1^2+\omega^2\tau_2^2) - \omega^2\mu_2((\omega^2+\mathcal{E}-1) + \mu_1[(\omega^2(\mathcal{E}-1)+1]|\Delta|^2(\omega^2\mu_1\mu_3[\omega^2(\mathcal{E}-1)+1]^2) - 2\omega^2\mu_1\mu_3(\omega^2+\mathcal{E}-1)^2 + \omega^2(\tau_1^2+\omega)(\omega^2+\mathcal{E}-1)\mu_1\mu_2\mu_3 > \omega^4(\omega^2+\mathcal{E}-1)(3[\omega^3(\mathcal{E}-1)+1]^2) - 2\omega^2(\omega^2+\mathcal{E}-1)^2\mu_1\mu_2\mu_3. \quad (4.3)$$

The inequality (4.3) will hold if we can show that the minimum of its left hand side is strictly greater than the maximum of its right hand side. The minimum of the left hand side of the inequality (4.3) is,

$$1-\mu_1 - \frac{\mu_1\tau_1}{4}\left[\frac{\mu_3}{\tau_3}+\frac{\mu_2}{\tau_2}\right],$$

While the maximum of the right hand side is $6(\mathcal{E}-1)\mu_1\mu_2\mu_3$. Thus the inequality (4.3) holds if, $(1-\mu_1) - \frac{\mu_1\tau_1}{4}\left[\frac{\mu_3}{\tau_3}+\frac{\mu_2}{\tau_2}\right] > 6(\mathcal{E}-1)\mu_1\mu_2\mu_3.$
This is possible by using Lemmas 1 and 2. The conclusion to the proof follows from the generalized theorem of Yacubovish.

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