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Periodic solutions of second order self-adjoint differential equations with variable potential

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ABSTRACT

The objective of this paper is concerned with the following second-order differential equation of the form $LY(t) = -(PY')'t) - r(t)P(t)Y'(t) + q(t) + Y(t) = f(t, Y(t)),\ t \in (0, \infty)$. By applying critical point theory, we establish sufficient conditions for the existence of periodic solutions of second order self-Adjoint differential equations with variable potentials.

Keywords: Periodic solutions; Self-Adjoint; variable potentials; boundary conditions; critical point theory.

INTRODUCTION

The problem of periodic solutions for differential equations has been the subject of many investigations. Researchers have used various techniques such as fixed point theory, critical point theory, coincidence degree theory, dynamical system theory etc. a series of existing results for periodic solutions have been obtained in the literature. We refer to [2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13], [2] and [13] consider the case with $r(t) \equiv 0$. When $f(t,x) \equiv 0$ for $(t,x) \epsilon$ R, as in (1.1) and (1.2).

In this paper, we consider the second-order differential equation with variable potentials given by
$$LY(t) = -(PY')'(t) - r(t)P(t)Y'(t) + q(t) + y(t) = f\big(t, y(t)\big). \quad t\epsilon(0, \infty) \tag{1.1}$$

With periodic boundary conditions.
$$Y(t) = Y(t+\omega), Y'(t) = Y'(t+\omega), t\epsilon(0, \infty) \tag{1.2}$$
Where $\omega > 0$ is the period and we assume the following;
$(H_1) = P, q, r\epsilon(\mathrm{R})$ with $\mathrm{P(t), q(t)} > 0$ and $\mathrm{r(t)} \geq 0$ for all $\mathrm{t} \epsilon\ (0, \infty)$;
$(H_2) = P(t) = P(t+\omega),\ q(t) = q(t+\omega),\ r(t) = r(\mathrm{t}+\omega)$ for all $\mathrm{t} \epsilon\ (0, \infty)$;
$(H_3) = f; (0, \infty).(0, \infty) \to (0, \infty)$ *is continuous with* $f(t, z) = f(t+\omega, z)$.

Equation (1.1) equivalent becomes the second order linear self adjoint differential equation.
$$LU(t) = \Delta[P(t)\Delta U(t-1) + q(t)u(t) = 0 \tag{1.3}$$

Equation (1.3) may arise from various fields such as electrical circuit analysis, matrix theory, control theory etc.

Our aim in this paper is to use critical point theory to establish the existence of nontrivial T. Periodic solutions of (1.3) into the existence of critical points of some functions called variational framework of (1.3). We shall prove the main results with Legget-Williams fixed point to establish the existence of at least one positive T-Periodic solution for (1.1) and (1.2).

Recall some basic notations and known results from critical point theory. Let H be a real Hilbert space, $K \in C'(H,R)$ which means that K is continuous Frechet-differentiable function defined on H. K is said to satisfy the Palais-small condition if and any sequence $\{x_n\}CH\ for\ which\ \{k(x_n)\}$ is bounded and $\{k'(x_n)\} \to 0\ as\ n \to \infty$ possesses a convergent subsequence in H. the following Lemma's were taken from [9] and will be useful in the proofs of non trivial T-Period of periodic solutions of (1.1).

LEMMA 1.1 (Mountain Pass Lemma)
Let H be a real Hilbert Space and assume that $K \in C'(H,R)$ satisfies the Palais-smale condition and the following condition holds;

(K_1), There exist constants ρ > 0 and a > 0 such that K(x) ≥ a, for all x ϵ $B_ρ$. where $B_\rho = \{x \in H: \|x\|11 < P\}$.
(K_2) (Ko) ≤ 0 and there exists $x_o \notin B_\rho\ \ such\ that\ k(x_o) \le 0$.
Then $C = infh\epsilon\ \Gamma\ sups\ \in [0,1]K\big(h(s)\big)is\ a\ positive\ critical\ value\ of\ K, where\ \Gamma = \{h \in c([0,1],H), h(0) = 0, h(1) = x_0\}$.

LEMMA 1.2 (Saddle Point Theorem)

Let H be a real Hilbert Space, $H = H_1 \oplus H_2, where\ H_1 \neq \{0\}$ and is finite dimentional. Suppose that $k \in c'(H,R)$, satisfies the Palais-Smale condition and the following holds.

(K_3), there exist constants $\sigma, \rho > 0\ such\ that\ ||B\rho n H_1 \le \sigma;$
(K_4), there is $e \in B\rho \cap H_1\ and\ a\ constant\ \omega > \sigma\ such\ that\ k|e + H_2 \ge \omega.$

Then K possesses a critical value c ≥ ω and $C = inf_{h\epsilon\Gamma} \max_{u \epsilon B\rho \cap H_1} K(h(u))$.

Where $\Gamma = \{h \in C(\overline{B_\rho} \cap H_1, H), |h|B_\rho \cap H_1 = id\}$.

2. VARIATIONAL FRAMEWORK FOR EQUATION (1.3)
Let S be the vector space of all real sequences of the form, $U = \{U(t)\}t \in R, t \in Z = (\dots., U(t), U(-t), U(-t+1), \dots\dots, U(-1), U(0), U(1), \dots. U(t), \dots), namely,$

$S = \{U = \{U(t)\}|U(t)| \in R, t \in Z\}$. Define the subset E of S as $E = \{U = \{U(t)\} \in S | U(t+T) = U(t), \forall t \in Z\}$.
Clearly, E is isomorphic to R^T. E can be equipped with the inner product.

$$\langle U, V\rangle E = \sum_{t=1}^{T} U(t)V(t) \text{ for any U, V, } \in E \tag{2.1}$$

by which the norm ||.||∈ can be induced by;

$$||.||\text{E} = \sqrt{\langle U, V\rangle E} = \left(\sum_{t=1}^{T} U^2\,(t)\right)^{\frac{1}{2}},\ \ U \in E \tag{2.2}$$

It is clear that E with the inner product (2.1) is a finite dimentional Hilbert Space and is linearly homeomorphic to R^T. Now define the function I on E as;

$$I(U) = \sum_{t=1}^{T}\left[\frac{1}{2}P(t)(\Delta U(t-1))^2 - \frac{1}{2}q(t)U^2 + F(t, U(t)\right], U \in E, \tag{2.3}$$

Where $F(t > O) = \int_0^x f(t,s)ds.\ Then\ I \in c'(E,R)$, and for any $U \in E,\ by\ using\ U(0) = U(T), U(I) = U(T+1)$ and (2.3), we can compute the Frechet derivative as;

$\frac{\partial I(U)}{\partial U(t)} = -\Delta[P(t)\Delta U(t-1)] - q(t)U(t) + f\big(t,u(t)\big),\ t \in Z(I,T).$

Thus, U is a critical point of I on E (i.e. $I'(U) = 0)$ $if\ and\ only\ if, \Delta[P(t)\Delta U(t-1)] + q(t)U(t) + f\big(t,u(t)\big)$, $\forall t \in\ Z(I,T)$. Which is precisely normalized equation (1.1) that gives equation (1.3). Therefore, we have reduced the existence of the non trivial T. Periodic solution of (1.1) to the existence of a critical point of I on E. in other words, the function I is just the variational frame work of (1.3). Where P and Q are written in matrix form as;

$$P = \begin{pmatrix} P(1)+P(2) & -P(2) & 0 & 0 & -P(1) \\ -P(2) & P(2)+P(3) & -P(3) & 0 & 0 \\ 0 & -P(3) & P(3)+P(4) & 0 & 0 \\ 0 & 0 & 0 & P(T-1)+P(T) & -P(T) \\ -P(1) & 0 & 0 & -P(T) & P(T)+P(1) \end{pmatrix}$$

$$Q = \begin{pmatrix} -q(1) & 0 & 0 & 0 & 0 \\ 0 & -q(2) & 0 & 0 & 0 \\ 0 & 0 & -q(3) & 0 & 0 \\ 0 & 0 & 0 & -q(T-1) & 0 \\ 0 & 0 & 0 & 0 & -q(T) \end{pmatrix}$$

3. TO PROVE THE NONTRIVIAL T-PERIOD SOLUTIONS OF EQUATION (1.1)

We assume the following;

(A_1) for each t ϵ Z $\lim_{|x|\to\infty} \frac{f(t,x)}{x} = 0.$ (3.1)

(A_2) There exist constants $a_1 > 0,\ a_2 > 0\ \ and\ \beta > Z\ \ such\ that\ \ \int_0^\infty f(t,s)ds \le -a_1|x|^\beta + a_2,\ \ \forall x \in R.$ (3.2)

By (A_2) $\lim_{|x|\to\infty} \frac{f(t,x)}{x} = -\infty.$

LEMMA 3.1: Suppose that $f\ \in C(Z \times R)$ satisfies (A_2); then I satisfies the Palais-Smale condition.

THEOREM 3.1: Suuppose that f satisfies (A_1) and (A_2). In addition, assume that the following holds.
$P(t) > 0\ \ for\ all\ t \in Z(I,T)$

$q(t) \le 0\ \ \ for\ all\ t \in Z(I,T)$ and there exists at least one t_o∈ Z (I,T) such that $q(t_o) < 0$. Then there exist at least two non trivial T-Periodic solution for (1.1).

PROOF:
We will use Lemma (1.1) to prove theorem (3.1). We need to verify that all the assumptions of the mountain pass theorem hold. The Palais-Smale condition started without proof in Lemma (3.1) will equally be tool for this theorem. Lemma (1.1) assumption can be demonstrated by matrix theory, which can easily be checked as P + Q is positive definite and its eigen values are represented by $\lambda_1, \lambda_2, \dots \dots, \lambda_T\ with\ 0 < \lambda_1 \le \lambda_2 \le \cdots \dots \le \lambda_T$. By ($A_1$), there exist $\rho\ >\ 0$ such that for any $|x| < \rho\ and\ t \in Z(I,T), F(t,x) \le \frac{1}{4}\lambda_1 x^2$. Thus, for any $u \in E, \big||U|\big| \le \rho, |U(t)| \le P, for\ all\ t \in Z(I,T), and\ I(u) \ge \frac{1}{2}\lambda_1||u||^2 - \frac{1}{4}\lambda_1||u||^2 = \frac{1}{4}\lambda_1||u||^2.$

Taking $a = \frac{1}{4}\lambda_1\rho^2 > 0,\ we\ have\ \ |u|\beta_\rho \ge a,$ and assuming (K_1) is verified, clearly, I(0) = 0.

For any given $\omega \in E\ with\ \big||\omega|\big| = 1\ and\ a\ constant\ \ \propto > 0,\ I(\propto \omega) = \frac{1}{2}\big((P+Q)\alpha\omega, \alpha\omega\big) + \sum_{t=1}^{T} f(t, \alpha\omega(t)) \le \frac{1}{2}\alpha^2\lambda_T - a_1\alpha^\beta \sum_{t=1}^{T}|\omega(t)|^\beta + a_2 T \le \frac{1}{2}\alpha^2\lambda_T - \frac{a_1 T(Z-\beta)}{2\alpha^\beta} + a_2 T \to -\infty\ as\ \ \propto \to +\infty.$

Thus we can easily choose a sufficiently large α such that α>ρ and for $U_o = \alpha\omega \in E,\ I(U_o) < 0$. Therefore, by Lemma (1.1), there exists at least one critical value $C \ge a > 0$. Suppose that $\overline{U}$ is a critical point corresponding to C, $i.e. I(\overline{U}) = C\ \ and\ \ I'(\overline{U}) = O$. By a similar argument to the omitted proof of lemma (3.1).

$I(U) \leq \frac{1}{2}|\lambda_{max}|||U||^2 E - a_1 T^{\frac{2-\beta}{2}}\left||U|\right|_E^{\beta} + a_2 T,\ \forall U \in E$ (3.3)

Thus one is bounded from above. We denote by C_{max} the supremum of $\{I(U),\ U \in E\}$. Since (3.2) implies that $\lim_{||u||\to+\infty} I(U) = -\infty$, -1 and +1 attains its maximum at some point $\overline{U}, i.e. I(\overline{U}) = C_{max.}\ Clearly, \overline{U} \neq 0$, then the proof of theorem (3.1) is completed.

REMARKS (3.1)
The periodic solutions we have obtained in the above proof are non trivial, but they may be non zero constant. If we want to obtain non constant periodic solutions, we only need to exclude non zero constant solutions.

COROLLARY (3.1)
Suppose that f satisfies (A_1), (A_2), (P) and (q) and $f(t, x) = 0\ for\ all\ t \in Z(I, T)$, if and only if x = 0. Then there exist at least two non constant T-Periodic solutions for (1.1).

4. PROOF OF THE MAIN RESULT – EXISTENCE OF A PERIODIC SOLUTION
We begin this section with necessary definitions and state a Lemma without proof and use extension of Leggett-Williams to establish the existence of solution (1.1) and (1.2).

Lemma (4.1)
Assume (H_1) and (H_2). Let h; $R \to [0, \infty]$ be continuous and let z, y be the solutions of (1.3). Then $Y(t) \geq \rho\left||Y|\right|\ for\ t\ \in (0, \omega), where\ the\ constant\ \rho\ is\ given\ by\ \frac{1}{exp\left\{\int_0^{\omega} \partial(j)dj\right\}\left(1+exp\int_0^{\omega}\delta(j)\delta j\right)}$.

DEFINITION (4.1)
A map β is said to be non negative continuous concave function on a cone P of a real Banach Space, E if $\beta: P \to [0, \infty]$ is continuous and $\beta(tx + (1-t)y \geq t\beta(x) + (1-t)\beta(y)$ for all x, y ∈ P and t ∈ (0, 1). Similarly, we say the map ∝ ia a negative non continuous convex function on a cone P of a real Banach space E if $\propto: P \to (0, \infty)$ is continuous and $\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y),\ for\ all\ x, y \in P\ and\ t \in (o, 1)$.

Let α and ψ be non negative continuous function on P and δ and Q be non negative continuous convex functions on P. Then, for non negative real numbers a, b, c and d, we define the following sets:

$A \coloneqq A(\alpha, \beta, a, d) = \{y \in P: a \leq \alpha(y)\ and\ \beta(y) \leq d\}$ (4.1)
$B \coloneqq B(\alpha, \delta, \beta, a, b, d) = \{y \in A: \delta(y) \leq b\}$ (4.2)
$C \coloneqq C(\alpha, \psi, \beta, a, c, d) = \{y \in A: C \leq \psi(y)\}$ (4.3)

We say that A is a functional wedge with concave functional boundary defined by the concave functional α and a convex functional boundary defined by the convex functional β. We say that an operator $T: A \to P$ is invariant with respect to the concave functional boundary, $if\ a\ \leq \alpha(Ty) for\ all\ Y \in A$, and that T is invariant with respect to the convex functional boundary, $if\ \beta(Ty) \leq d\ for\ all\ Y \in A$. Note that A is a convex set. The following theorem is an extension of [11].

THEOREM (4.1) [EXTENSION OF LEGGETT-WILLIAMS]
Suppose P is a cone in a real Banach space E, $\alpha\ and\ \psi$ are non negative continuous concave functions on P, $\delta\ and\ \beta$ are non negative continuous convex function on P, and for non negative real numbers a, b, c and d, the sets A, B and C are as defined in (4.1), (4.2) and (4.3) respectively. Furthermore, suppose that A is a bounded set of P, that $T: A \to P$ is completely continuous and that the following conditions hold;

(A) $\{Y \in A: C < \psi(y)\ and\ \delta(y) < b\} \neq 0\ \ and\ \ \{Y \in P: \alpha(y) < a\ and\ \ d < \beta(y)\} = \emptyset$;
(B) $\alpha(Ty) \geq a\ for\ all\ y \in B$;
(C) $\alpha(Ty) \geq a\ for\ all\ y \in A\ with\ \delta(Ty) > b$;
(D) $\beta(Ty) \leq d\ for\ all\ y \in C; and$
(E) $\beta(Ty) \leq d\ for\ all\ y \in A\ with\ \psi(Ty) < c$.

Then T has a fixed point $Y^* \in A$.

The following cone structure will have helped to apply this theorem to equations (1.1) and (1.2). Let E denote Banach space $\in (0,\infty)$ with the supremum norm

$$||Y|| = sup|Y(t)|,\ t \in |0,\omega| \qquad (4.4)$$

And for ρ given in Lemma (4.1), define the cone P ⊂ E via;

$$P \coloneqq \{Y \in E: Y(t) \geq \rho||Y||, Y(t+\omega) = Y(t)\ and\ Y'(t+\omega) = Y'(t)\ \ for\ all\ t \in (0,\infty)\} \qquad (4.5)$$

Then the fixed points of the integral operator $T: P \to E$ given by (4.6)

$$Ty(t) \coloneqq \int_t^{t+\omega} G(t,s) f(s, y(s)) ds \qquad (4.6)$$

Are solutions of (1.1) and (1.2). Using (4.5), let the non negative continuous concave function $\psi: P \to (0,\infty)$ be defined by

$$\psi(y) = \min_{t\in(0,\omega)} Y(t),\ \ Y \in P \qquad (4.7)$$

$for\ Y \in P,\ \ 0 < \psi(y) \leq ||Y|$ by Lemma (4.1). Furthermore, let the linear equations $\alpha, \beta: P \to (o,\infty)$ be defined by

$$\beta(y) = \alpha(y) = \frac{1}{\omega}\int_0^{\omega} Y(t)dt \qquad (4.8)$$

And $\delta: P \to (0,\infty) be\ defined\ by\ \delta(y) = ||Y||$, making δ a non negative continuous convex function.

We shall use the Leggett-Williams fixed point to prove the existence of at least one positive solution to (1.1) and (1.2) in the following theorem.

THEOREM (4.2)

Assume (H_1) – (H_3) for any d > 0, suppose the following hold:

(a) $f(t,y) \geq m_1 y + b1\ \ for\ Y \in [\rho^3 d, \rho d], for\ all\ t \in [0,\omega]$;

(b) $f(t,y) \leq m_1(ld) + b1\ \ for\ Y \in [\rho^2 d, d/\rho], for\ all\ t \in [0,\omega]$,

Where for $\rho, g_*\ and\ g^*$ given in Lemma (4.1), we have taken

$$m_1 = \frac{1}{2g^*\omega} > 0\ \ and\ \ b_1 = \frac{\rho^2 d(1-\frac{1}{2}\rho)}{\omega g^*} > 0 \qquad (4.9)$$

Then the operator T has at least one positive solution $Y^* \in A(\alpha, \beta, \rho^2, d, d)$, and thus equation (1.1) and (1.2) has at least one positive solution.

PROOF:

For any d > 0 and ρ in Lemma (4.1), $let\ a = c = \rho^2 d, b = \rho d$. By the properties of G(t, s) given in Lemma (4.1), we have $T: A(\alpha, \beta, a, d) \to P$. Arzela-Ascoli theorem shows that T is a completely continuous operator from the properties of G and f, and by the definition of β, we have that A is a bounded subset of the cone P. If $Y \in P\ and\ \beta(y) > d, then\ \alpha(y) = \beta(y) > d > \rho^2 d = a.$ Therefore, $\{Y \in P: \alpha(y) < a\ and\ d < \beta(y) = \Phi\}$. We define a constant function $Yo = \frac{1}{2}(c+b), then\ \alpha(y_o) = y_o = \frac{1}{2}(a+b) \geq a\ \ and\ \beta(y_o) = y_o = \frac{1}{2}(\rho^2 d + \rho d) \leq d\ since\ \rho \in (0,1), putting\ y_o \in A. Moreover, \psi(y_o) = y_o > c, \delta(y_o) = y_o < b, and\ thus\ \{Y \in A: C < \psi(y)\ and\ \delta(y) < b\} \neq \Phi.$

CONDITION 1:

$\alpha(Ty) \geq a,\ \ for\ all\ Y \in \beta$. For any $Y \in B, there\ is\ a\ t_o \in (o,\omega)\ such\ that\ y(t_o) = \alpha(y) \geq a.$ It follows that for $Y \in B$,

$\rho d = b \geq \delta(y) = ||Y|| \geq \alpha(y) = y(t_o) \geq \psi(y) \geq \rho||Y|| \geq \rho\alpha(y) \geq \rho a = \rho^3 d,$ and therefore $\psi(y) \geq \rho a = \rho^3 d$. Thus, for all $Y \in B\ we\ have\ \rho^3 d \leq Y(t) \leq \rho d\ for\ all\ t \in (o,\infty)$. Consequently by condition 1, $\alpha(TY) = \frac{1}{\omega}\int_0^{\omega}\int_t^{t+\omega} G(t,s) f(s,y(s)) dsdt \geq$

$\frac{g_*}{\omega}\int_o^{\omega}\int_t^{t+\omega} f(s,y(s)) dsdt = \frac{g_*}{\omega}\int_o^{\omega}\int_0^{\omega} f(s,y(s)) dsdt = g_*\int_o^{\omega} f(s,y(s)) ds \geq g_*\int_o^{\omega}(m_1 y(s) + b_1) ds =$

$g_*\omega(m_1\alpha(y) + b_1) \geq g_*\omega(m_1 a + b_1) = a,\ \ using\ a = \rho^2 d\ and$ (4.9).

CONDITION 2:
$\alpha(Ty) \geq a$ *for all* $Y \in A$ *with* $\delta(Ty) > b$. *Let* $Y \in A$ *with* $\delta(Ty) > b$. *Thus Lemma* (4.1), $Ty(S) \geq \rho b$ *for all* $S \in (o, \omega)$, *so that* $\alpha(Ty) = \frac{1}{\omega}\int_0^\omega Ty(S)ds \geq \frac{1}{\omega}\int_0^\omega \rho b ds = \rho b = a$.

COMDITION 3:
$\beta(Ty) \leq d$, *for all* $Y \in C$. *If* $Y \in C$, *then for all* $t \in (0, \infty)$,
We have,
$C \leq \psi(y) \leq Y(t) \leq \frac{\psi(y)}{\rho} \leq \frac{\beta(y)}{\rho} \leq \frac{d}{e}$.
Hence for all $t \in (0, \infty)$, *we have* $\rho^2 d = c \leq Y(t) \leq \frac{d}{\rho}$, so that by condition 2, we see that,
$\beta(TY) = \frac{1}{\omega}\int_0^\omega \int_t^{t+\omega} G(t,s) f(s, Y(s)) dsdt \leq \frac{g^*}{\omega}\int_0^\omega \int_t^{t+\omega} f(s, Y(s)) dsdt = \frac{g^*}{\omega}\int_0^\omega \int_0^\omega f(s, Y(s)) dsdt =$
$g^* \int_0^\omega f(S, Y(S)) ds \leq g^* \int_0^\omega (m_1(\rho d) + b_1) ds = g^*\, \omega(m_1(\rho d) + b_1) \leq g^* \omega(m_1 d + b_1) = \frac{1}{2} d(1 + 2\rho - \rho^2) \leq d$.
Since $\rho \in (0,1)$,

COMDITION 4:
$\beta(TY) \leq d$, *for all* $Y \in A$ *with* $\psi(TY) < C$.
Let $Y \in A$ *with* $\psi(TY) < C$. Then using Lemma (4.1), we have;
$Ty(S) \leq \frac{c}{\rho}$ *for all* $S \in (0, \omega)$, *hence* $\beta(Ty) = \frac{1}{\omega}\int_0^\omega Ty(S)ds \leq \frac{1}{\omega}\int_0^\omega \frac{c}{\rho} ds = \frac{c}{\rho} = \rho d < d$.

Therefore, the hypotheses of theorem (4.1) have been satisfied. Thus the operator T has at least one positive solution,
$Y^* \in A(\alpha, \beta, a, d) = A(\alpha, \beta, \rho^2 d, d)$.

This completes the proof.

REFERENCES

[1] Pankov, A. and Zakharchenko, N. **2001**. *Acta Appl. Math*. 65, 295 – 303.
[2] Anderson, D.R. **2005**. *Non linear Anal*, 60, 1, 101 – 115.
[3] Atici, F.M., Cabada, A. and Otero-Espinar, V. **2003**. *J. Differential Equ. Appl*., 9, 765 – 775.
[4] Graef, J.R., Kong, L. and Wang, H. **2008**. *J. Differential Equations*, 245, 1185 – 1197.
[5] Butler, G.J. **1980**. *SIAM J. Math. Anal. II*, 190 – 200.
[6] Hale, J.K. and Mawhin, J. **1974**. *J. Differential Equations*, 15, 295 – 307.
[7] Mawhin, J. and Willem, M. **1989**. Critical point theory and Hamiltonian systems, Springer, New York.
[8] Torres, P. **2003**. *J. Differential Equations*, 190, 643 – 662.
[9] Rabinowitz, P.H. **1986**. Mini, max methods in critical point theory with applications to differential equations, CBMS Issues, Math. Ed., 65.
[10] Avery, R.I., Henderson, J. and Anderson, D.R. **2009**. *Comm. Appl. Nonlinear Anal*, 16, 4, 39 – 44.
[11] Ma, S.W. Wang, Z.C. and Yu, J.S. **1998**. *J. Differential Equations*, 145, 274 – 294.
[12] Li, X. and Zhang, J. **2009**. *J. Equations Appl*., 15, 6.
[13] Liu, Y. Ge, W. and Gui, Z. **2005**. *Anal. Appl*., 3, 2, 145 – 155.
[14] Guo, Z.M., Yu, J.S. and Zuo, X. **2005**. *J. London Math. Soc*., 71, 146 – 160.