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Propagation of Errors in Euler Methods

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Abstract

The error bound of a numerical algorithm is very crucial to it's selection for use in computatio of numerical values of Initial Value Problems. In this work, we investigate and compute the error bounds for the new Euler scheme proposed by Abraham in [1]. We compare and contrast this same parameter for the existing Euler Methods and the new proposed method.
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INTRODUCTION

Ordinary Differential Equations (ODEs) are of basic importance in engineering mathematics because many physical laws and relations appear mathematically in the form of a ODE. Whenever an initial condition is attached to an ODE, it becomes an Initial Value Problem (IVP). The number of instances where an exact solution of an ODE can be found by analytical means is very limited.

Therefore, one of the objectives of numerical analysis is to solve such complex problems using only the simple operations of arithmetic, to develop and evaluate methods for computing numerical results. The method of computing the numerical values are called algorithm. In the search for good algorithms, error bounds for the methods become very crucial because it determines the choice of step length, and of course, the speed in generating the numerical results.

Generalizations of Euler Methods

Given a function $f\big(x, y\,(x)\big)$ and an "initial value" $y(x_0)$ corresponding to a solution value at(x_0), we seek to evaluate numerically the function $y(x)$satisfying

$$\left.\begin{array}{l} y'(x) = f\big(x, y\,(x)\big),\ x \in [x_0, x_{end}], \\ y(x_0) = y_0 \end{array}\right\} \qquad (1)$$

An approximate solution to an Initial Value Problem (IVP) (1) is typically obtained by iterating a set of *difference equations* that approximate the original system.
The famous method of Euler was published in his three volume work *Institutiones Calculi Integrals* in the years 1768 to 1770, republished in his collected works in 1913 [4]. It involves computing a discrete set$\{y_n\}$, for arguments $\{x_n\}$, using the difference equation

$$EM: \; y_{n+1} - y_n \begin{cases} = \Phi_E(x_n, y_n; h) \\ = hf(x_n, y_n), n = 1,2,\dots,m \end{cases} \quad (2)$$

where the step size $h = x_{n+1} - x_n$
The Euler method is simple. It uses only one piece of information from the past and evaluates the driving function only once per step. However, it is not practical for computational purposes since a considerable effort is required to improve accuracy. In spite of $\Phi_E(x_n, y_n; h)$ its limitations, the Euler method is the fundamental building block for the higher accuracy methods, be it Runge–Kutta or Linear Multistep methods [8].

Since the difference equation is linear in y_n and f_n, and being a one-step method, it can easily handle IVPs that require variable step length. Since Euler proposed his historical Euler method in 1768, there has been lot of developments on this class of method. Among others, Abraham [1], recently, proposed a new improvement on Euler Method, which is called Modified Improved Modified Euler Method. In this work, we examine the error bound for this newly proposed algorithm in relation to the other existing Euler methods. Our computation show that the order of accuracy of the method is 2, when applied to Initial Value Problem, the method competes well with the existing methods. However, we discovered that for certain step length, the method did not yield good results. The summary of these achievements [1, 7] are presented in table 1

Table 1: Development of *Euler Methods*

Method	$y_{n+1} - y_n = \Phi_E(x_n, y_n; h)$	Stability Function $R_{Method}(z)$
EM	$= hf(x_n, y_n)$	$1 + z$
ME	$= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right)$	$1 + z + \frac{1}{2}z^2$
IE	$= \frac{1}{2}h\left(f(x_n, y_n) + f\left(x_n + h, y_n + hf(x_n, y_n)\right)\right)$	$1 + z + \frac{1}{2}z^2 + \frac{1}{4}z^3$
IME	$= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf\left(x_n + h, y_n + hf(x_n, y_n)\right)\right)$	$1 + z + \frac{1}{2}z^2 + \frac{1}{2}z^3$
MIME	$= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right)\right)$	$1 + z + \frac{1}{2}z^2 + \frac{1}{4}z^3$

Propagation of Errors in Euler Methods
Euler methods, like other one-step methods are based on the principle of discretization. These methods have the common feature that no attempt is made to approximate the exact solution $y(x)$ over a continuous range of the independent variable. Approximate values are sought only on a set of discrete points x_0, x_1, x_2 ... We denote the true solution of the differential equation at $x = x_n$ by $y(x_n)$, and the appropriate solution obtained by applying any of the Euler methods as y_n. We wish to investigate the propagation of error of these methods which is a crucial property of the method. This study also helps in the selection of steplength and thus, the speed of generating numerical results for IVPs.

Definition 3.0.1. *The Local Truncation Error (LTE) at x_{n+1} of the Euler methods is defined to be τ_{n+1} where*

$$\tau_{n+1} = y(x_{n+1}) - y(x_n) - \Phi_{Method}(y\,(x_n); h) \qquad (3)$$

and $y(x)$ is the theoretical solution of the IVP (1)

If we make the localizing assumption that no previous errors have been made (that is , that $y_n = y(x_n)$, then the LTE of *Euler methods* satisfies

$$\tau_{n+1} = y(x_{n+1}) - y_{n+1}$$

The study of error bounds also plays a significant role in the design of program codes for solving IVP (1). Only few codes control the LTE committed at every integration step by demanding that [5],

$$\tau_{n+1} \leq h_n \tau_n \qquad (4)$$

where $h_n = x_{n+1} - x_n$ is the current stepsize and τ_n is the allowable error tolerance, which may depend on the independent variable x_n. Most practical codes, however, replace h_n on the righthand side of (4) by unity, thus adopting error per step criterion [5].

A user is actually interested in the true or global error specified by

$$e_{n+1} = y(x_{n+1}) - y_{n+1}, \qquad (5)$$

This global truncation error e_{n+1} is defined such that it is no longer assumed that no previous truncation errors have been made. And it is well known that the variational equation

$$e'(x) = J\,\big(x, y\,(x)\big)\,e\,(x)\,, e\,(a) = \delta \qquad (6)$$

(where J is the Jacobian matrix associated with the IVP (1) says how an error δ at $x = a$ propagates. The approximate equation (6) is satisfied by the error-neglecting second order terms [5].

The propagation of errors depends on two factors namely:

- the local error and
- the nature/stability of the problem

For instance if the IVP (1) is inherently stable (that is, all the eigenvalues of *J* have negative real parts), then the local errors may damp out with increasing x; otherwise, the errors will be magnified with increasing *x* [7]. Bulirsh and Stoer [2], constructed asymptotic upper and lower bounds on the global errors emanating from extrapolation methods to IVPs. Shampine [10], generalized this idea for any one-step method endowed with an asymptotically correct LTE estimator.

There exist fundamental obstacles in the direct control of the global error, but in recent years, appreciable progress has been attained and reliable estimation of the global error [5].
The LTE and the roundo errors constitute a sequence of perturbations that shift computed solutions to the neighbouring integral curves.

The stability of a discretization method for (1) demands that, provided the starting global error

$e_0 \neq 0$, the ultimate global error e_n should be bounded [5]; that is a finite constant K exists such that

$$e_n < Ke_0$$

Therefore, the LTE for *MIME method* can be obtained by using the Taylor's series expansion for $y(x_{n+1})$ and (y_{n+1}) from table 1 as follows,

$$\tau_{n+1} = y_n + hf + \frac{1}{2}h^2 f f_y + \frac{1}{6}h^3(f f_y^2 + f^2 f_{yy})$$

$$-y_n + hf + \frac{1}{2}h^2 f f_y + \frac{1}{8}h^3\left(2\left(f f_y^2 + f^2 f_{yy}\right) + O(h^4)\right) \quad (7)$$

$$= \frac{1}{24}h^3\left(f^2 f_{yy} - 2 f f_y^2\right) + O(h^4) \quad (8)$$

This can also be written as

$$\tau_{n+1} = \varphi\left(y(x_n)\right)h^3 + O(h^4) \quad (9)$$

where

$$\varphi\left(y(x_n)\right) = \frac{1}{24}\left(f^2 f_{yy} - 2 f f_y^2\right) \quad (10)$$

is called the principal error function, and $\varphi\left(y(x_n)\right)h^3$ the Principal Local Truncation Error (PLTE) [6, 5].

Suppose the following bounds for f and its partial derivatives hold for $x \in [a, b], y \in (-\infty, \infty)$

$$|f(y)| \leq Q, \left|\frac{\partial^j f(y)}{\partial y^j}\right| < \frac{P^j}{Q^{j-i}}, j \leq p \quad (11)$$

where P and Q are positive constants [9], and p is the order of the method (in this case p = 2). Then,

$$|fy| < P, |fyy| < P^2 Q^{-1} \text{ and } |fyyy| < P^3 Q^{-2}$$

and the following bounds on the errors are obtained for the *MIME and Euler Methods* in table 1:

Table 2: Principal Error functions and Bounds of *Euler Methods*

Method	Principal Local Truncation Error (PLTE)	*Region of Absolute Stability*
EM	$\left\|\varphi_{EM}\left(y(x_n)\right)h^2\right\| \leq \frac{1}{2}h^2 QP$	$-2 < z < 0$
ME	$\left\|\varphi_{ME}\left(y(x_n)\right)h^3\right\| \leq \frac{1}{3}h^3 QP^2$	$-2 < z < 0$
IE	$\left\|\varphi_{IE}\left(y(x_n)\right)h^3\right\| \leq \frac{1}{12}h^3 QP^2$	$-2 < z < 0$
IME	$\left\|\varphi_{IME}\left(y(x_n)\right)h^3\right\| \leq \frac{1}{3}h^3 QP^2$	$-1.47797 < z < 0$
MIME	$\left\|\varphi_{MIME}\left(y(x_n)\right)h^3\right\| \leq \frac{1}{12}h^3 QP^2$	$-2 < z < 0$

Numerical Computations

In this section we compute the appropriate meshsize bounds so as to integrate the IVPs in Examples 1 - 4 [5, 6], using the Euler methods stated in the table with an allowable error tolerance $\varepsilon = 10^{-4}$. These computations are shown in figure 1. We also implement these schemes on the IVPs in the given examples. The numerical results generated are also plotted and displayed in figures 2 – 9.

Example 1

Consider the IVP,

$$y'(x) = -10(y(x)^2 - 1), y(0) = 2, 0 \leq x \leq 1 \qquad (12)$$

of which its bounds can readily be established as $Q = 10, P = 20$. The theoretical solution of this problem is given as

$$y(x) = 1 + \frac{1}{10x + 1}$$

The graphs of the numerical values of $y(x)$ generated for this example using the Euler methods are displayed in figures 2 and 3.

Example 2

Consider the general test problem,

$$y'(x) = y(x), \quad y(0) = 1, \qquad 0 \leq x \leq 1 \qquad (13)$$

of which its bounds can readily be established as $Q \ exp(1), P = 1$. The theoretical solution of this problem is given as

$$y(x) = exp(x)$$

The graphs of the numerical values of $y(x)$ generated for this example using the Euler methods are displayed in figures 4 and 5.

Example 3

Consider the IVP,

$$y'(x) = \sqrt{y(x)}, \quad y(0) = 1, \ 0 \leq x \leq 1 \qquad (14)$$

of which its bounds can readily be established as $Q = 1, P = \frac{1}{2}$. The theoretical solution of this problem is given as

$$y(x) = \frac{1}{4}(x + 2)^2$$

The graphs of the numerical values of $y(x)$ generated for this example using the Euler methods are displayed in figures 6 and 7.

Example 4

Consider the IVP,

$$y'(x) = 1 + (y(x))^2, y(0) = 1, 0 \leq x \leq \frac{\pi}{4} \qquad (15)$$

of which its bounds can readily be established as $Q = 2, P = 2$. The theoretical solution of this problem is given as

$$y(x) = \tan\left(x + \frac{\pi}{4}\right)$$

The graphs of the numerical values of $y(x)$ generated for this example using the Euler methods are displayed in figures 8 and 9.

CONCLUSION

The comparison between MIME method with other existing Euler methods shows that, MIME and Improved Euler methods have the same stepsize bound, stability function and Region of Absolute Stability (see table 1, 2 and figure 1). From figures 2 – 9, it is obvious that the newly proposed MIME metho d has a higher order of accuracy than many existing Euler methods. However, the results obtained in some cases were not as accurate as some existing Euler method.
For instance, despite the fact that, the stability functions, region of absolute stability and stepsize bounds are the same for MIME and IE methods:

- In example 1, for $h = 0.05$, MIME generated the best results from figure 2, whereas from figure 3 for $h = 0.1$, IE method gave the best results

- In example 2, for h = 0.05and h = 0.05, MIME generated the best results from figure 4 and 5 respectively

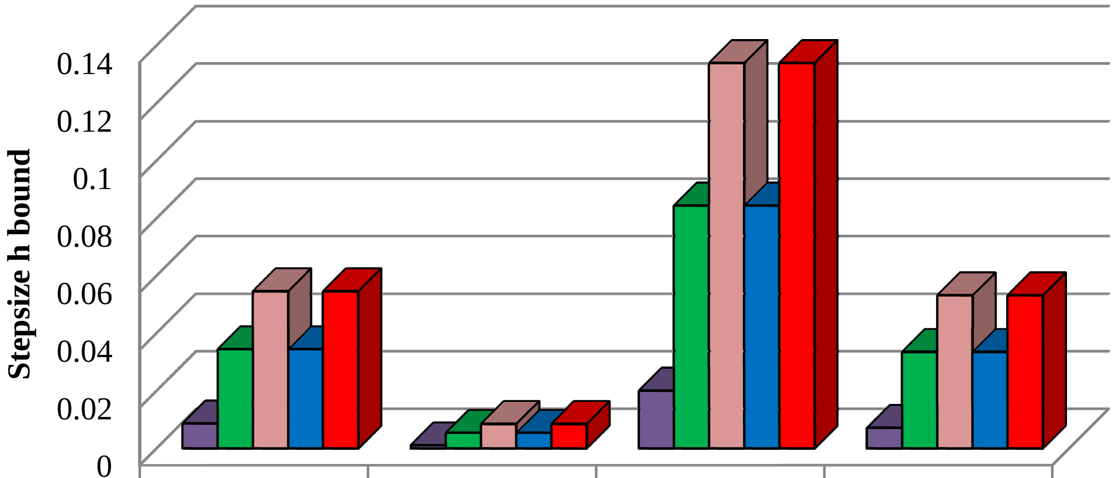
- In example 3, MIME method performed best for h = 0.05and h = 0.05 from figures 6 and 7 respectively.

- Also, in example 4, MIME method performed best for h = 0.05from figure 8, whereas all the methods did not yield good results for h = 0.05 from figure 9. This confirms the fact that stability of the problem being solved also plays significant role in generating a good result.

We therefore conclude that the study of propagation of errors in Euler methods is quite significant and should find relevance in the development of program codes for solving IVPs in ODEs among other relevance.

In addition, we have also displayed the stability regions of these *Euler methods* in the complex plane in Figures 10 and 11. This analysis will also find relevance in the selection of a good numerical scheme for solving Initial Value Problems in Ordinary Differential Equations, especially when methods with low computational costs are of major interest.

Figure 1: Stepsize (h) bounds for Euler Methods



	Example 1	Example 2	Example 3	Example 4
EM	0.008577639	0.001	0.02	0.007071068
ME	0.034369834	0.005313293	0.084343267	0.033471648
IE	0.05455871	0.008434327	0.13388659	0.053132928
IME	0.034369834	0.005313293	0.084343267	0.033471648
MIME	0.05455871	0.008434327	0.13388659	0.053132928

Figure 2: Graph of Numerical Values of y(x) using h = 0.05 (0 ≤ x ≤ 1)

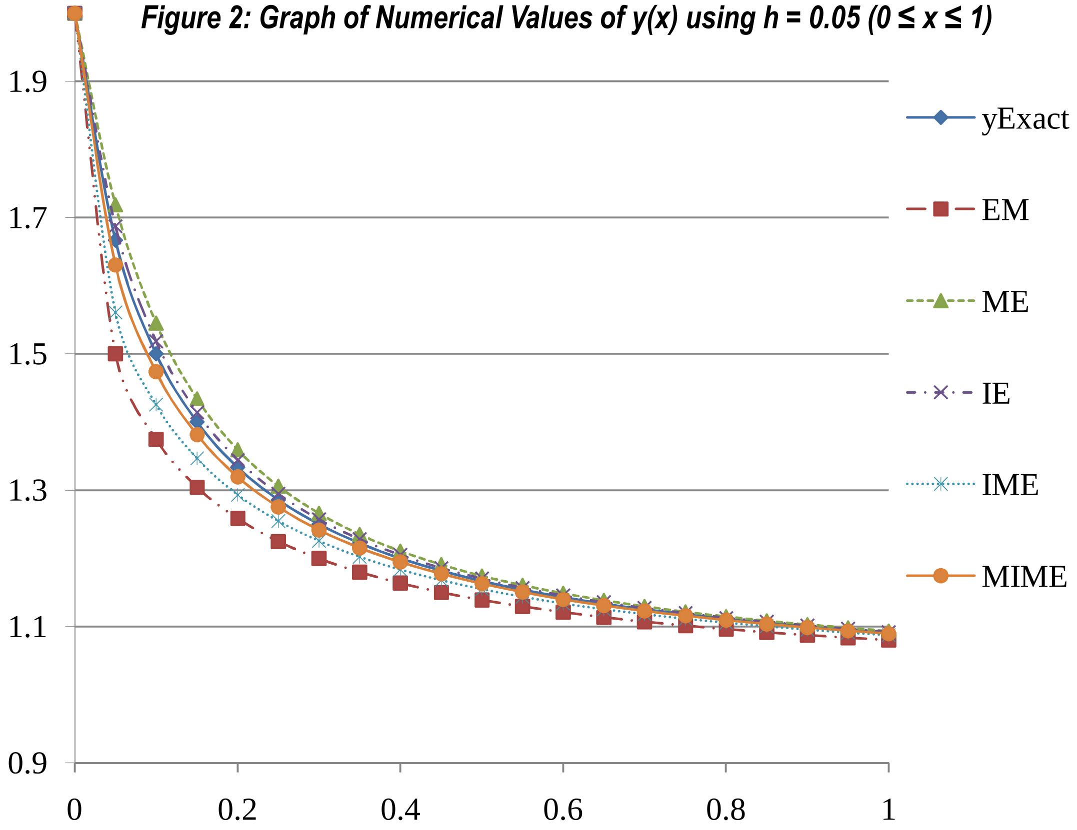


Figure 3: Graph of Numerical Values of y(x) using h = 0.1 (0 ≤ x ≤ 1)

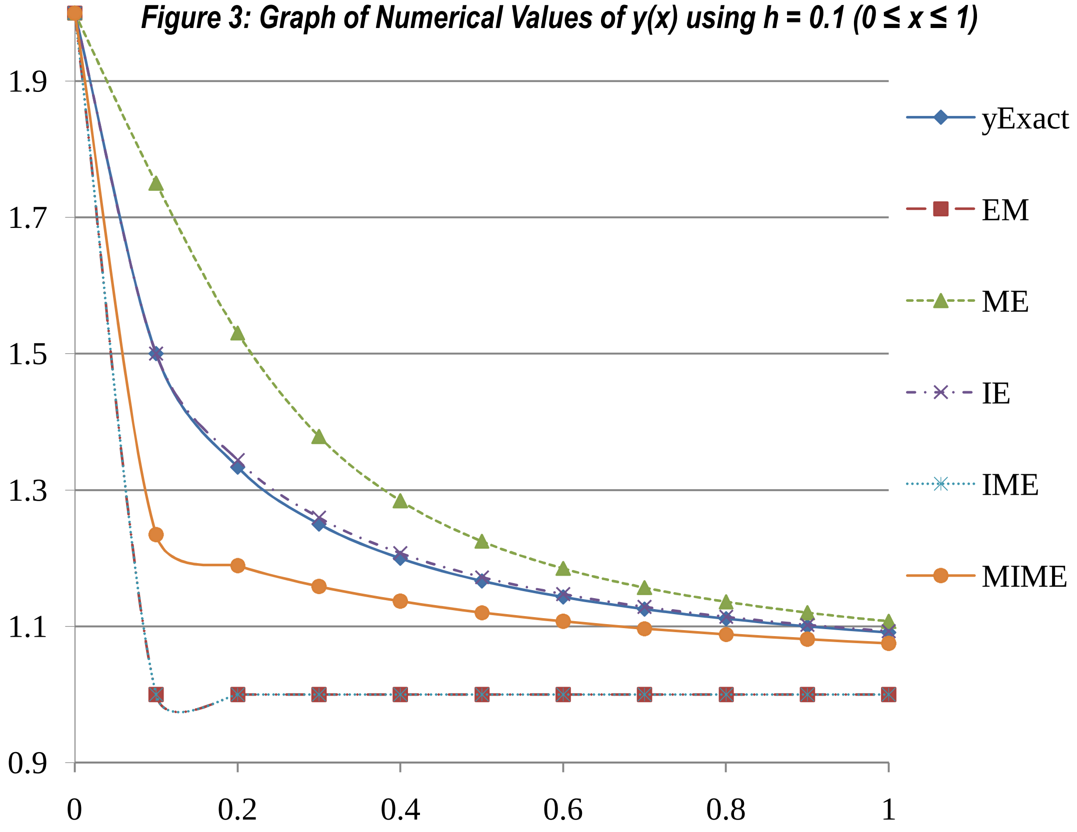


Figure 4: Graph of Numerical Values of y(x) using h = 0.05 (0 ≤ x ≤ 0.7)

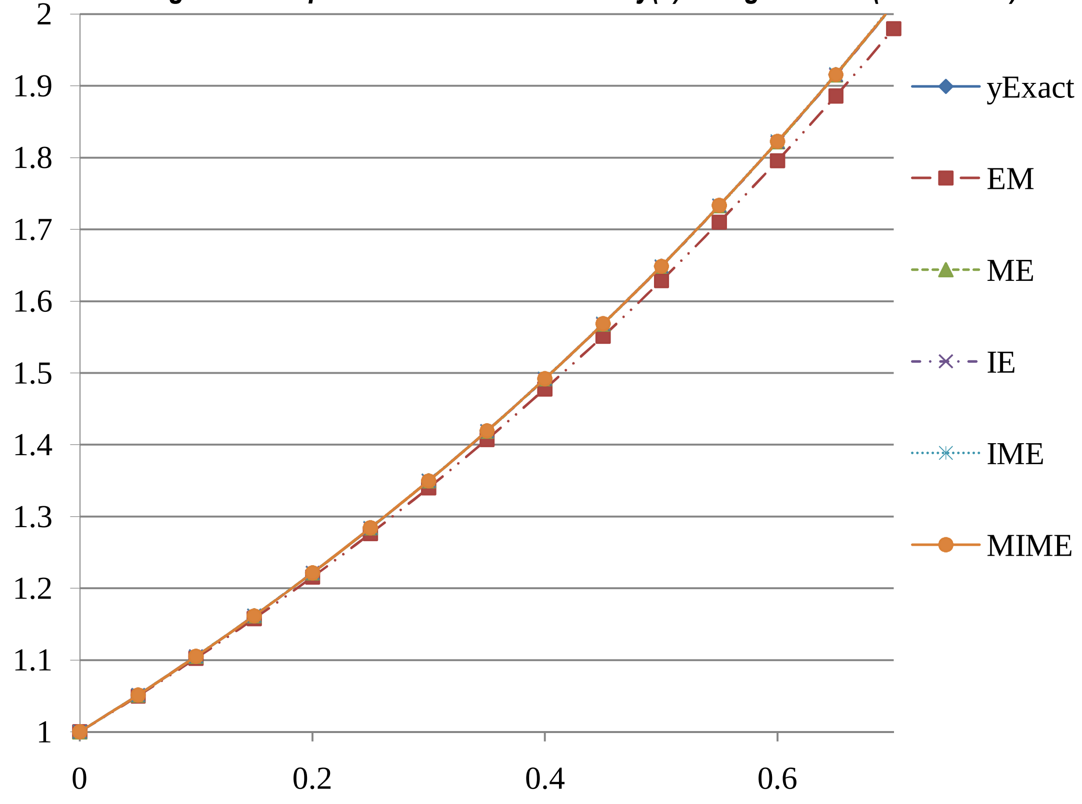


Figure 5: Graph of Numerical Values of y(x) using h = 0.5 (0 ≤ x ≤ 0.7)

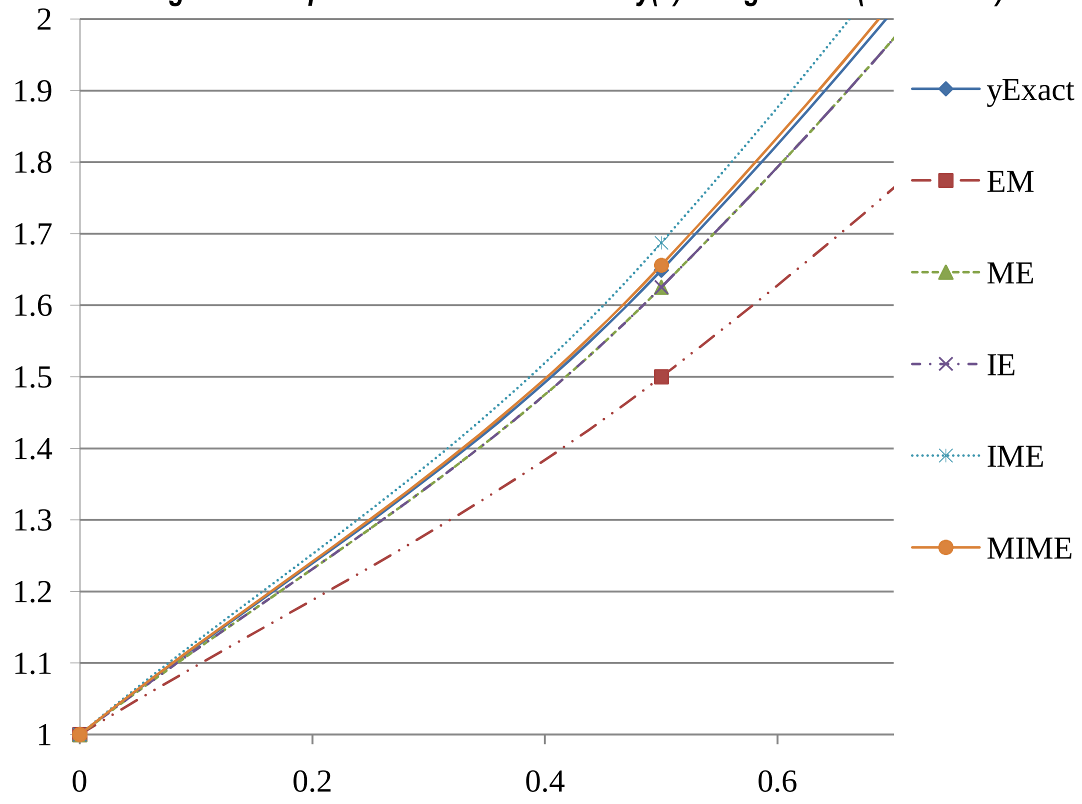


Figure 6: Graph of Numerical Values of y(x) using h = 0.1 (0 ≤ x ≤ 0.7)

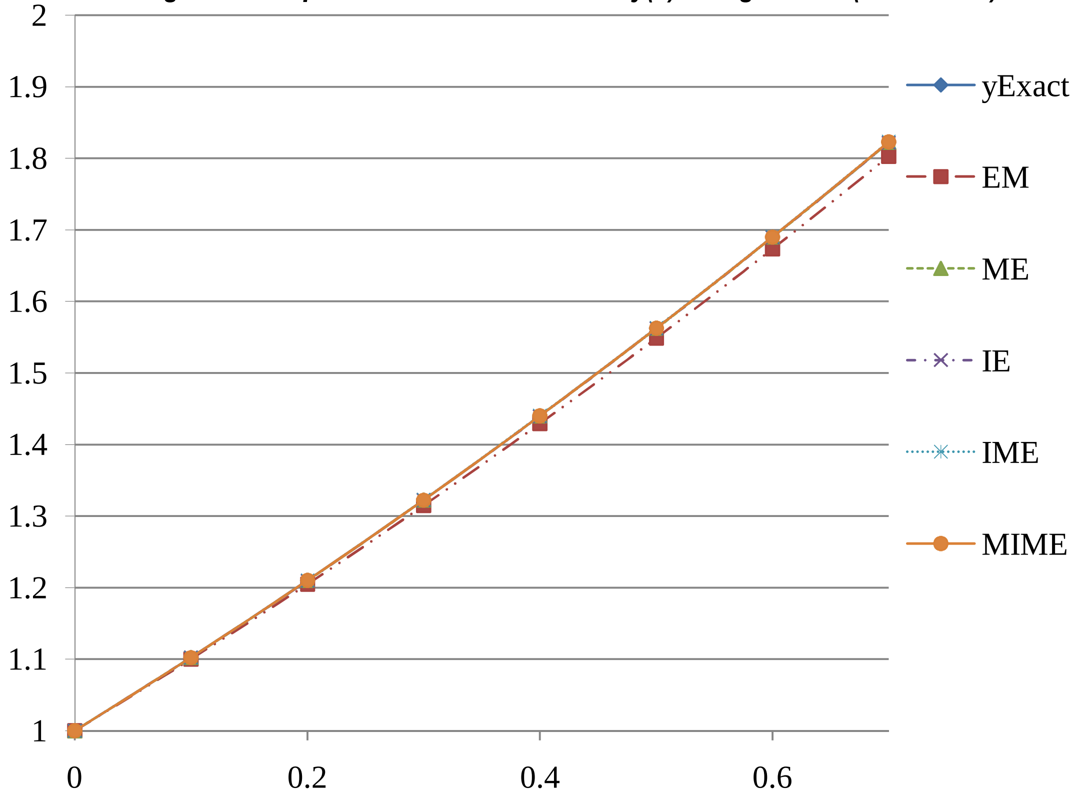


Figure 7: Graph of Numerical Values of y(x) using h = 0.5 (0 ≤ x ≤ 0.7)

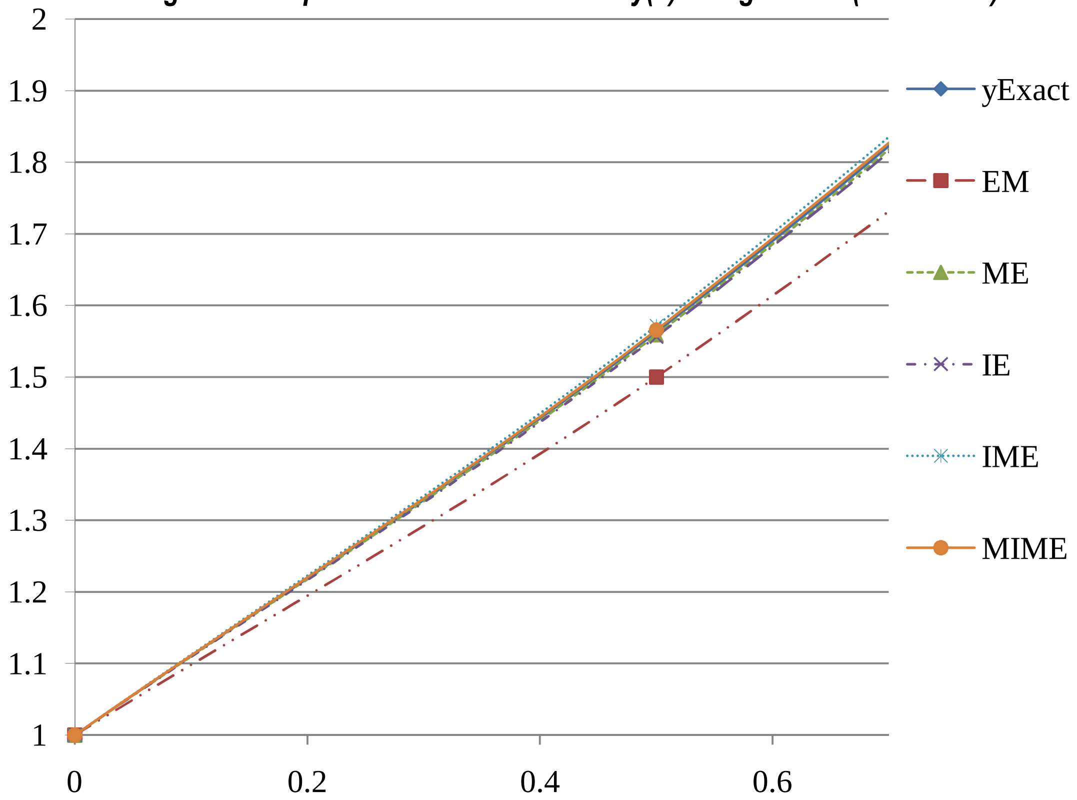


Figure 8: Graph of Numerical Values of y(x) using h = 0.1 (0 ≤ x ≤ 0.7)

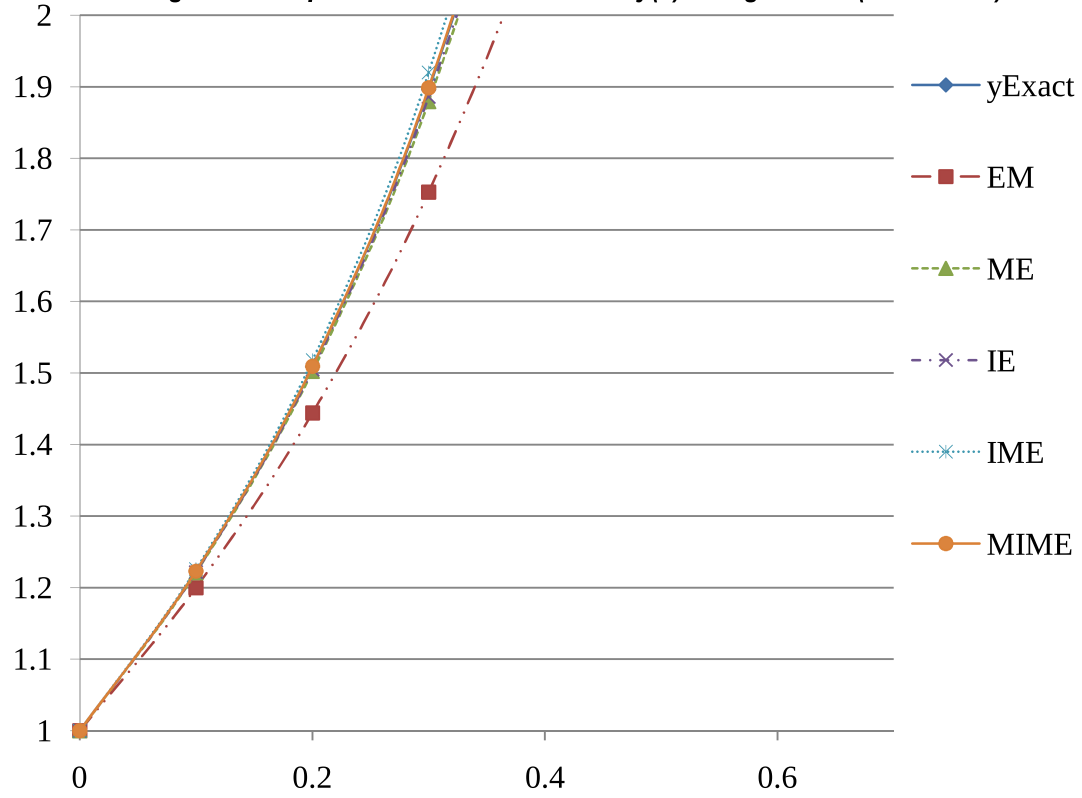
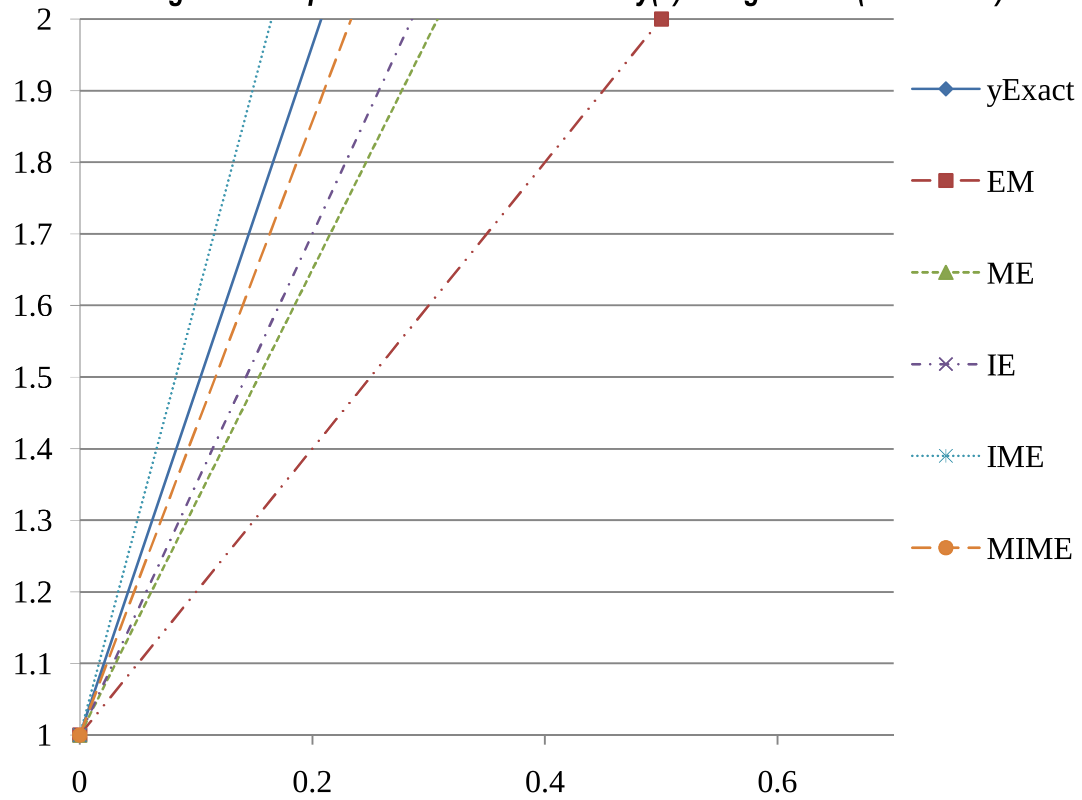


Figure 9: Graph of Numerical Values of y(x) using h = 0.5 (0 ≤ x ≤ 0.7)



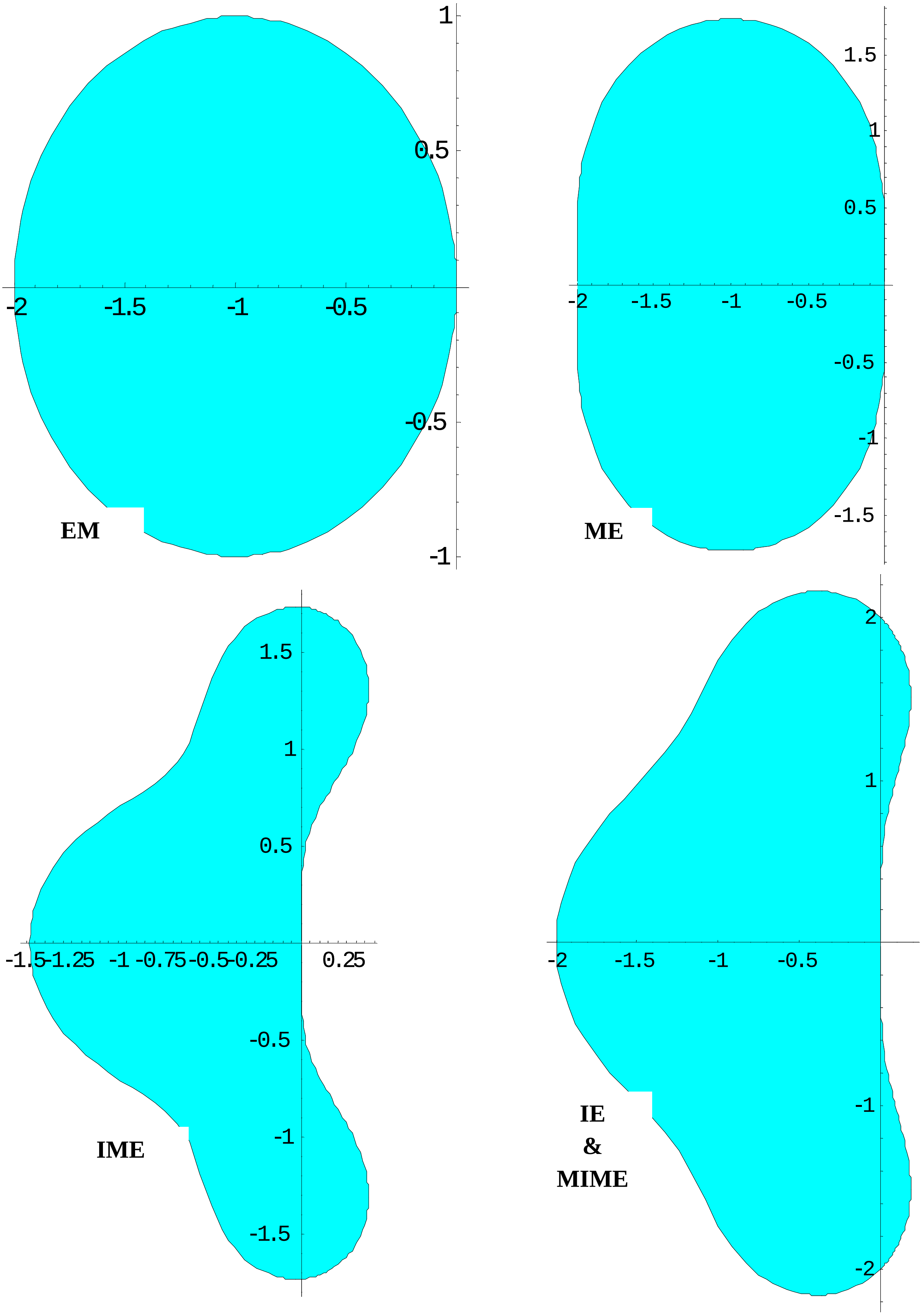


Figure 10: The Stability regions of the Euler Methods in the Complex Plane

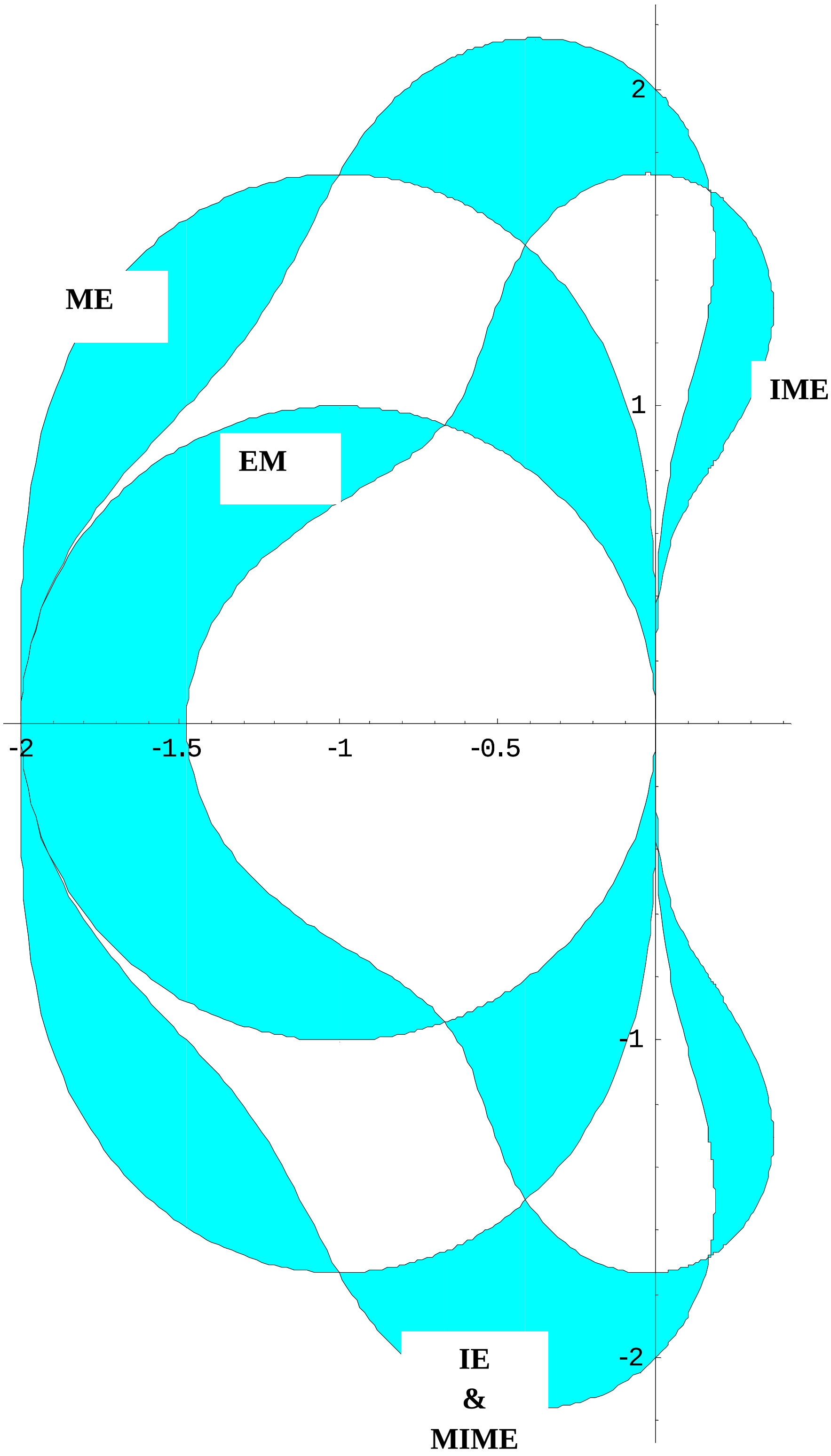


Figure 11: Comparison of Stability regions for the Euler Methods (EM, ME, IME, IE and MIME).

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