Representation of p-Groups by character tables

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ABSTRACT

In this paper, we applied some group concepts to construct some p-groups as well as to display their nature, which represent them by character tables. It was observed that if \( g \) and \( g^{-1} \) belongs to different conjugacy class say \( C_i \) and \( C_j \) then the entries in the character table for \( C_j \) are complex conjugate of the corresponding entries for \( C_i \). Also, if \( g \) and \( g^{-1} \) belongs to the same conjugacy class \( C_i \) then the entries in the character table for \( C_i \) are real valued. We also applied the Groups, Algorithms and Programming (GAP) version 4.4.12 to assist towards validations of result.

INTRODUCTION

Representation theory is concerned with the ways of writing a group as a group of matrices. Not only is the theory beautiful in its own right, but also provides one of the keys to a proper understanding of finite groups. For example, it is often vital to have a concrete description of a particular group; this is achieved by finding a representation of the group as a group of matrices.

2. Preliminaries

To begin, with there is the need for some preliminary fact and brief discussion of notations. Some of them can be verified quite readily.

2.1 Definition

Let \( A \) \( G_1 \) and \( G_2 \) be groups. A group homomorphism is a function \( \phi: G_1 \rightarrow G_2 \) such that for all \( a, b \in G_1 \), \( \phi(ab) = \phi(a) \cdot \phi(b) \) and \( f(a^{-1}) = (f(a))^{-1} \).

An invertible group homomorphism is called group isomorphism.

2.2 Definition

A representation of a group \( G \) with representation space \( V \) is a homomorphism \( \rho: g \rightarrow \rho(g) \) of \( G \) into \( GL(V) \).

From the homomorphism property we have for \( g, h \in G \):
\[ v_\rho(gh) = v_\rho(g)\rho(h), \]
\[ v_\rho(1) = v_{1_\nu} \]

### 2.3 Definition

Two representations \( \rho, \varphi: G \to GL(n, F) \) are said to be equivalent if there exist a \( n \times n \) matrix \( P \) over \( F \) such that

\[ P^{-1}(g)P = \varphi(g) \quad \text{for all} \quad g \in G \]

### 2.4 Theorem

Let \( \rho(g) \) be a matrix representation of \( G \). Then the character \( \chi(g) \) of \( \rho \) has the following properties

(i) Equivalent representations have the same character.

(ii) If \( g \) and \( h \) are conjugate in \( G \), then \( \chi(g) = \chi(h) \).

*Proof:* (i) If \( \varphi(g) \) and \( \rho(g) \) are equivalent representations, then by a well-known result, \( \varphi(g) \) and \( \rho(g) \) have the same characteristic matrix. Thus the respective coefficient of \( \lambda^{m-1} \) are equal

\[ b_{11}(g) + b_{22}(g) + \ldots + b_{nn}(g) = a_{11}(g) + a_{22}(g) + \ldots + a_{nn}(g) \]

Hence equivalent representations have the same character.

(i) Suppose that \( g \) and \( h \) are conjugate elements via \( t \) in \( G \). Then \( h = t^{-1}gt \).

Thus in any matrix representation \( \rho(g) \) we have

\[ \rho(h) = \rho(t^{-1}gt) = \rho(t^{-1})\rho(g)\rho(t), \quad \text{since } \rho \text{ is a representation.} \]

Identifying \( \rho(t) \) with \( T \) in \( \varphi(g) = T^{-1}\rho(g)T \) we find that \( \rho(h) \) and \( \rho(g) \) are equivalent representation. Hence by (i)

\[ tr\rho(h) = tr\rho(g) \text{ ie } \chi(g) = \chi(h). \]

### 2.5 Definition

Let \( \rho: G \to GL(n, F) \) be a representation of a group \( G \) over a field \( F \). The function \( \chi: G \to F \) defined by \( \chi(g) = tr(\rho(g)) \) is called character of \( \rho \).

The character satisfies the following properties:

1. \( \chi_\rho(e) = \deg(\rho) \).
2. \( \chi_\rho(xgx^{-1}) = \chi_\rho(g) \forall x, g \in G. \)
3. \( \chi_\rho(g^{-1}) = \chi_\rho(g) \)

### 2.6 Particular type of characters:

(i) Trivial character: The character corresponding to the trivial representation of \( G \) is called trivial character of \( G \). This character has value 1 over all elements of \( G \), \( \chi(g) = 1, \forall g \in G \).

(ii) Linear Character: A character of degree 1 is called a linear character. The first step in constructing character table of \( G \), which carries complete description of the structure of the group \( G \), is often writing down all the linear characters.

(iii) Permutation character: A permutation character \( \chi_\pi \) is the character afforded by a permutation representation \( \pi: G \to S_n \).

### 2.7 Lemma

Suppose \( F = \mathbb{C} \) and let \( \chi \) be character of \( G \). Then \( \chi(g^{-1}) = \overline{\chi(g)} \forall g \in G \), where \( \overline{\chi(g)} \) denote the complex conjugate of \( \chi(g) \).

### 2.8 Definition

Let \( \chi \) and \( \psi \) be characters of \( G \). Then inner product is defined as

\[ \langle \chi, \psi \rangle = |G|^{-1} \sum_{g \in G} \chi(g)\psi(g^{-1}) \]

Since summing over all $g \in G$ is the same as summing over all $g^{-1} \in G$, we have $\langle \chi, \psi \rangle = \langle \psi, \chi \rangle$. Also by Lemma 2.7
\[ \langle \chi, \chi \rangle = |G|^{-1} \sum_{g \in G} \chi(g) \overline{\chi(g)} = |G|^{-1} \sum_{g \in G} |\chi(g)|^2 > 0 \]
\[ \langle \chi, \psi \rangle = |G|^{-1} \sum_{g \in G} \chi(g) \overline{\psi(g)} = |G|^{-1} \sum_{g \in G} (g^{-1}) \overline{\psi(g)}(g) = \langle \psi, \chi \rangle = \langle \chi, \psi \rangle. \]

Hence $\langle \chi, \psi \rangle \in \mathbb{R}$.

2.9 Theorem
Let $\chi$ and $\psi$ be characters of two non-isomorphic irreducible representations of $G$. Then we have
(i) $\langle \chi, \psi \rangle = 0$,
(ii) $\langle \chi, \chi \rangle = 1$

2.10 Corollary
If $\chi_1, \ldots, \chi_k$ are all the irreducible characters of $G$ and $\chi = \sum_{i=1}^{k} n_i \chi_i$ and $\psi = \sum_{i=1}^{k} m_i \chi_i$
are any two characters of $G$, then
$\langle \chi, \phi \rangle = \sum_{i=1}^{k} n_i m_i$.

2.11 Lemma
Let $\chi$ be character of $G$. Then $\chi$ is irreducible if and only if $\langle \chi, \chi \rangle = 1$. This is a criterion for the irreducibility of a character.

Proof:
$(\Rightarrow)$ This was proved in theorem 2.9
$(\Leftarrow)$ If $\chi_1, \ldots, \chi_k$ are all irreducible characters of $G$, we can express as
$\chi = \sum_{i=1}^{k} n_i \chi_i$ and $\psi = \sum_{i=1}^{k} m_i \chi_i$.
Assume that $\langle \chi, \chi \rangle = 1$, the by corollary 2.11
$\langle \chi, \chi \rangle = \sum_{i=1}^{k} n_i^2 = 1$. Since $n_i \in \mathbb{Z}$ and $n_i \geq 0$ we have that for one $i$, $n_i = 1$ and for all $j \neq i$, $n_j = 0$, and so $\chi = \chi_i$ and $\chi$ is irreducible.

For the next two lemmas we recall the character $\chi_R$ of the regular representation of $G$. For $F = \mathbb{C}$ and $g \in G$:
$\chi_R(g) = \begin{cases} |G|, & \text{if } g = 1 \\ 0, & \text{otherwise} \end{cases}$

2.12 Corollary
Let $r_1, r_2, \ldots, r_k$ be the irreducible representations of $G$. Then
$\|G\| = \sum n_i^2$
Where $n_i$ is the dimension of the representation $r_i$.

2.13 Corollary
Let $\chi_1$ be the character of the trivial representation. The $G$ is simple iff
$\ker \chi_i = 1$ for $2 \leq i \leq k$.

2.14 Lemma
If $\chi$ is a character of $G$, then $\ker \chi = \{ g \in G \mid \chi(g) = \chi(1) \}$.

2.15 Theorem
A group $G$ is abelian if and only if every irreducible character $\chi_i$ of $G$ is linear.
2.16 Theorem
Let \( \chi_1, \ldots, \chi_k \) be all irreducible characters of \( G \) and let \( g_1, \ldots, g_k \) be the representatives of the conjugacy classes \( C_1, \ldots, C_k \) of \( G \).
Then we have
(i) The row orthogonality relation
\[
\sum_{\alpha=1}^{k} \frac{\chi_i(g_\alpha)\chi_j(g_\alpha)}{\chi_\alpha(g_\alpha)} = \delta_{ij} \quad \forall \ i, j = 1, \ldots, k;
\]
(ii) The column orthogonality relation
\[
\sum_{\alpha=1}^{k} \chi_i(g_\alpha)\chi_j(g_\alpha) = \delta_{\alpha\beta}|\chi_\alpha(g_\alpha)| \quad \forall \ \alpha, \beta = 1, \ldots, k,
\]
where we again define \( \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \).

2.17 Theorem
The number of irreducible characters of a group \( G \) is equal to the number of conjugacy classes of \( G \).

2.18 Lemma
If \( \chi_1, \ldots, \chi_k \) are all the irreducible characters of \( G \), then
\[
\sum_{i=1}^{k} \chi_i^2(1) = |G|.
\]
Proof: \( |G| = \chi_R(1) = \left[ \sum_{i=1}^{k} \chi_i(1) \chi_i \right](1) = \sum_{i=1}^{k} \chi_i^2(1) \).

RESULTS

Illustration 3.1
Consider the p-group \( G_1 \)
\[
G_1 = \{(1), (13), (24), (12)(34), (13)(24), (14)(23), (1234), (1432)\}
\]
Now, \( |G_1| = 8 = 2^3 \), which is a p-group.
The conjugacy classes of \( G_2 \) are:
\[
C_1 = (1), \quad C_2 = (12), \quad C_3 = (34), \quad C_4 = (1234), \quad C_5 = (1432).
\]
It will be observed from the conjugacy classes of \( C_5 \) above, for any \( g \in G \) \( g \) and \( g^{-1} \) belong to the same conjugacy class.

According to By theorem 2.17, there are five irreducible representations for the group \( G_1 \). By corollary 2.12, we find a set of five positive integers, \( l_1, l_2, l_3, l_4 \) and \( l_5 \) which satisfy the equation \( l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 = 8 \). The only values of \( l_i \) \( (i = 1, \ldots, 5) \) which satisfy this requirement are 1, 1, 1, 1 and 2. Thus, the group \( G_1 \) has four

1-dimensional irreducible representations and one 2-dimensional irreducible representation. Set \( \chi_1 \) to be the trivial character: \( \chi_1(1) = 1 \quad \forall \ g \in G_1 \). That is by Lemma 2.18, in any group, there will be 1-dimensional representations whose character are all equal to 1, since
\[
\sum_R(\chi_1(R))^2 = (1)1^2 + (2)1^2 + (1)1^2 + (2)1^2 + (2)1^2 = 8.
\]
The other representations will have to be such that
\[
\sum_R(\chi_i(R))^2 = 8
\]
which can be true if and only if each \( \chi_i(R) = \pm 1 \). By Theorem 2.16, each of the other three representations has to be orthogonal to the first irreducible representation, \( \chi_1 \).
Thus, there will have to be two +1’s and two -1’s. The fifth representation will be of dimension 2, hence $\chi_5(C_1) = 2$. In order to find out the values of $\chi_5(C_2), \chi_5(C_3), \chi_5(C_4)$ and $\chi_5(C_5)$, the orthogonality relationships in Theorem 2.17 will be used:

Partial character table of $G_1$:

<table>
<thead>
<tr>
<th>Rep $g$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>C_g</td>
<td>$</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>e</td>
</tr>
</tbody>
</table>

Orthogonality of $\chi_5$ with the other known $\chi_i$’s we obtained the following set of equations.

\[ \langle \chi_5, \chi_1 \rangle = 2 + 2a + 2b + c + 2d = 0 \quad \text{(3.1)} \]
\[ \langle \chi_5, \chi_1 \rangle = 2 + 2a + 2b + c - 2d = 0 \quad \text{(3.2)} \]
\[ \langle \chi_5, \chi_1 \rangle = 2 + 2a + 2b + c - 2d = 0 \quad \text{(3.3)} \]
\[ \langle \chi_5, \chi_1 \rangle = 2 + 2a - 2b + c + 2d = 0 \quad \text{(3.4)} \]

Adding (3.1) and (3.2), (3.3) and (3.4) we got

\[ 2 + 2a + c = 0 \quad \text{(3.5)} \]
\[ 2 - 2a + c = 0 \quad \text{(3.6)} \]

Solving (3.5) and (3.6) we have $a = 0, c = -1$

Substituting the values of $a$ and $c$ into equations (3.1) and (3.2) and solving it, we have

\[ b = d = 0 \]

Complete character table of $G_2$:

<table>
<thead>
<tr>
<th>Rep $g$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>C_g</td>
<td>$</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>0</td>
</tr>
</tbody>
</table>

The characters of $G_1$ are all irreducible since by theorem 2.12, that is

\[ \langle \chi_1, \chi_1 \rangle = \langle \chi_2, \chi_2 \rangle = \langle \chi_3, \chi_3 \rangle = \langle \chi_4, \chi_4 \rangle = \langle \chi_5, \chi_5 \rangle = 1 \]

Also by 2.19, $\sum_{i=1}^{k} \chi_i^2(1) = |G_1|$ and by 2.9 $\langle \chi_1, \chi_1 \rangle = 0$

With theorem 2.15, we see that $G_1$ is not abelian, since one of its irreducible representation, $\chi_5$ has degree 2 ≠ 1. Also by Lemma 2.14, we have

\[ ker \chi_1 = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 = |G_3|, \quad ker \chi_2 = C_1 \cup C_2 \cup C_4, \quad ker \chi_3 = C_1 \cup C_3 \cup C_4, \]

\[ ker \chi_4 = C_1 \cup C_2 \cup C_5 \quad \text{and} \quad ker \chi_5 = C_1. \]

Thus the kernel of $\chi_2$ is a cyclic group of order 4. By corollary 2.13, $ker \chi_2 \neq 1$, and, $G_1$ is not simple. From the character table of $G_1$ above, the entries of characters of $C_5$ on table are real.
Illustration 3.2
Consider the group:
$$G_2 = \{(1),(1,5)(2,6)(3,7)(4,8),(1,8,7,6,5,4,3,2),(1,4,7,2,5,8,3,6),$$
$$\,(1,6,3,8,5,2,7,4),(1,2,3,4,5,6,7,8),(2,8)(3,7)(4,6),(1,5)(2,4)(6,8),(1,7)(2,6)(3,5),$$
$$\,(1,3)(4,8)(5,7),(1,8)(2,7)(3,6)(4,5),(1,4)(2,3)(5,8)(6,7),(1,6)(2,5)(3,4)(7,8),$$
$$\,(1,2)(3,8)(4,7)(5,6)\}$$

Now, $$|G_2| = 16 = 2^4$$, which is a p-group.
$$G_2$$ has seven conjugacy classes, namely
$$C_1 = \{ 1 \}$$, $$C_2 = \{ (1357) (2468) \}$$, $$C_3 = \{ (1753) (2864) \}$$,
$$C_4 = \{ (12345678) \}$$, $$C_5 = \{ (18765432) \}$$,

We noticed above, $$C_2, C_3$$ and $$C_4$$ are conjugate to their inverses.
By theorem 2.17, $$G_2$$ has seven irreducible representations.
$$|G_2| = 1^2 + 1^2 + 1^2 + 2^2 + 2^2 + 2^2 = 16$$ by corollary 2.12.
Set $$\chi_1$$ to be the trivial character: $$\chi_1(1) = 1 \forall \; g \in G_2$$.
Set $$\chi_2$$ to be a sign character: $$\chi_2(g) = 1$$ if $$g$$ is even permutation, and $$\chi_2(g) = -1$$ if $$g$$ is odd.
We will use theorem 2.16 to obtain all the linear characters of $$G_2$$. $$\chi_5$$ has dimension 2, it must be orthogonal to $$\chi_1, \chi_2, \chi_3$$ and $$\chi_4$$. Therefore one of the characters must be -2.

The partial character table of $$G_2$$ is:

<table>
<thead>
<tr>
<th>Rep</th>
<th>$$C_1$$</th>
<th>$$C_2$$</th>
<th>$$C_3$$</th>
<th>$$C_4$$</th>
<th>$$C_5$$</th>
<th>$$C_6$$</th>
<th>$$C_7$$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$$</td>
<td>C_g</td>
<td>$$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$$\chi_1$$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$$\chi_2$$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$$\chi_3$$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$$\chi_4$$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$$\chi_5$$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$$\chi_6$$</td>
<td>2</td>
<td>$$a_1$$</td>
<td>$$b_1$$</td>
<td>$$c_1$$</td>
<td>$$d_1$$</td>
<td>$$e_1$$</td>
<td>$$f_1$$</td>
</tr>
<tr>
<td>$$\chi_7$$</td>
<td>2</td>
<td>$$a_2$$</td>
<td>$$b_2$$</td>
<td>$$c_2$$</td>
<td>$$d_2$$</td>
<td>$$e_2$$</td>
<td>$$f_2$$</td>
</tr>
</tbody>
</table>

Now by orthogonality relation theorem 2.16, we obtained the entries of the rows 6 and 7 as follows:

Using the orthogonality of the first and second columns we got
$$1.1 + 1.(-1) + 1.1 + 1.(-1) + 2.0 + 2a_1 + 2a_1 = 0$$
$$a_1 + a_1 = 0$$ (3.13)
$$1.1 + (-1).(-1)+1.1+(-1)+0.0+a_1, \; a_2 = \frac{16}{2} = 8$$
$$a_1^2 + a_2^2 = 4$$ (3.14)
Solving (3.13) and (3.14) we have
$$a = -\sqrt{2}$$ and $$b = \sqrt{2}$$
Similarly, the orthogonality of first and fifth columns
$$d_1 + d_2 = 0$$ (3.15)
$$d_1^2 + d_2^2 = 8$$ (3.16)
Solving (3.15) and (3.16) we have
$$d_1 = -2, \; d_2 = 2$$

Orthogonality of first and sixth columns
Using theorem 2.15, we see that
\[ e_1 + e_2 = 0 \]  
(3.17)  
\[ e_1^2 + e_2^2 = 0 \]  
(3.18)  
Solving (3.17) and (3.18) we have \( e_1 = e_2 = 0 \)
The orthogonality of the first and seventh columns
\[ f_1 + f_2 = 0 \]  
(3.19)  
\[ f_1^2 + f_2^2 = 0 \]  
(3.20)  
Solving (3.19) and (3.20) we have, \( f_1 = f_2 = 0 \)
Using the orthogonality of first and third columns  
\[ 2 + b_1 + b_2 = 0 \]  
(3.21)  
\[ b_1^2 + b_2^2 = 4 \]  
(3.22)  
Solving (3.21) and (3.22) we have \( b_1 = 0, b_2 = -2 \)
The orthogonality of first and fourth columns
\[ c_1 + c_2 = 0 \]  
(3.23)  
\[ c_1^2 + c_2^2 = 4 \]  
(3.24)  
Solving (3.23) and (3.24) we have \( c_1 = \sqrt{2}, c_2 = -\sqrt{2} \)

This complete the character table of \( G_2 \)

<table>
<thead>
<tr>
<th>( \text{Rep} g )</th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
<th>( C_6 )</th>
<th>( C_7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>C_g</td>
<td>)</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_3 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_4 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_5 )</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_6 )</td>
<td>2</td>
<td>-\sqrt{2}</td>
<td>0</td>
<td>\sqrt{2}</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_7 )</td>
<td>2</td>
<td>\sqrt{2}</td>
<td>-2</td>
<td>-\sqrt{2}</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Using theorem 2.15, we see that \( G_2 \) is not abelian, since three of its irreducible representation, \( \chi_3, \chi_6 \) and \( \chi_7 \) has degree \( 2 \neq 1 \). Also by Lemma 2.14, we have \( ker \chi_2 = C_1 \cup C_3 \cup C_5 \cup C_6 \). Thus the kernel of \( \chi_2 \) is a cyclic group of order 8. Thus \( ker \chi_2 \neq 1 \), and by corollary 2.13, \( G_2 \) is not simple. We observed from character table of \( G_2 \) that the entries of the characters of \( C_2, C_3 \) and \( C_4 \) are real.

**Illustration 3.3**
Consider the group:
\[ G_3 = \{ (1, (1.5)(2,6),(3,7)(4,8),(1,3)(2,4),(1,4,3,2),(2,1,3,4),(1,7,3,5),(2,8,4,6),(1,8,4,7,3,6,2,5), (1,6,2,7,3,8,4,5),(5,7)(6,8),(1,2,4),(5,7,6,8),(1,4,3,2),(5,7,6,8),(1,2,3,4),(5,7,6,8),(1,5,3,7,2,6),(1,7,8,3,6,4,5),(1,8,2,5,3,6,4,7),(1,6,4,5,3,8,2,7),(5,8,7,6),(1,3,2,4),(5,8,7,6),(1,2,3,4),(5,8,7,6),(1,5,4,8,3,7,2,6),(1,7,2,8,3,5,4,6),(1,8,3,6),(2,5,4,7),(1,6,2,7),(3,8),(4,5),(5,6,7,8),(1,3,2,4),(5,6,7,8),(1,2,3,4),(5,6,7,8),(1,5,2,3,6,7,4,8),(1,7,4,6,3,5,2,8), (1,8,2,5,3,6,4,7),(1,6,3,8),(2,7,4,5) \} \]
\[ |G_3| = 32 = 2^5 \], which is a p-group.
The group \( G_3 \) has fourteen conjugacy classes and is represented by:
\[ C_1 = \{ 1 \} \quad C_2 = \{ (1,2,3,4),(5,6,7,8) \} \quad C_3 = \{ (1,3),(2,4),(5,7),(6,8) \} \]
\[ C_4 = \{ (1,4,3,2),(5,8,7,6) \} \quad C_5 = \{ (1,2,3,4),(5,6,7,8) \} \]
\[ C_6 = \{ (1,2,3,4),(5,7,6,8) \} \quad C_7 = \{ (1,2,3,4),(5,8,7,6) \} \]
\[ C_8 = \{ (1,3,2,4),(5,7,6,8) \} \quad C_9 = \{ (1,3,2,4),(5,8,7,6) \} \]
\[ C_{10} = \{ (1,4,3,2),(5,8,7,6) \} \quad C_{11} = \{ (1,5)(2,6),(3,7)(4,8),(1,6,2,7),(3,8),(4,5),(1,7,2,8),(3,5),(4,6),(1,8,2,5,3,6,4,7) \} \]
\[ C_{12} = \{(1,5,2,6,3,7,4,8),(1,6,2,7,3,8,4,5),(1,7,2,8,3,5,4,6),(1,8,2,5,3,6,4,7)\} \]
\[ C_{13} = \{(1,5,3,7,2,6),(1,6,3,8,2,7),(1,7,3,5,2,8),(1,8,3,6,2,5)\} \]
\[ C_{14} = \{(1,5,4,8,3,7,2,6),(1,6,4,5,3,8,2,7),(1,7,4,6,3,5,2,8),(1,8,4,7,3,6,2,5)\} \]

The conjugacy classes of \( G_3 \), above show that that there is an element \( g \) for which \( g \) and \( g^{-1} \) belong to the different conjugacy classes.

\[ (C_2, C_5), (C_5, C_{10}), (C_6, C_9) \) and \( (C_{12}, C_{14}) \).

Correspondingly by theorem 2.17 \( G_3 \) have fourteen irreducible representations. From the relation in corollary 2.12,

\[ |G_3| = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 2^2 + 2^2 + 2^2 + 2^2 + 2^2 = 32 \]

Therefore, \( G_3 \) must have exactly eight in equivalent one-dimensional and six two-dimensional irreducible representations.

Since \( \chi_1(1) \), gives the dimension of the representation, may therefore fill out the first column in the character table. Following the same procedure as in illustration 3.1 and theorem 2.16, we obtained the complete character table of \( G_3 \) as:

**Character table of \( G_3 \):**

<table>
<thead>
<tr>
<th>Rep g</th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
<th>( C_6 )</th>
<th>( C_7 )</th>
<th>( C_8 )</th>
<th>( C_9 )</th>
<th>( C_{10} )</th>
<th>( C_{11} )</th>
<th>( C_{12} )</th>
<th>( C_{13} )</th>
<th>( C_{14} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>G_3</td>
<td>)</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_3 )</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_4 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_5 )</td>
<td>1</td>
<td>-i</td>
<td>-i</td>
<td>1</td>
<td>-i</td>
<td>1</td>
<td>1</td>
<td>-i</td>
<td>1</td>
<td>-i</td>
<td>1</td>
<td>1</td>
<td>-i</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_6 )</td>
<td>1</td>
<td>i</td>
<td>-i</td>
<td>1</td>
<td>i</td>
<td>-i</td>
<td>1</td>
<td>i</td>
<td>-i</td>
<td>1</td>
<td>i</td>
<td>-i</td>
<td>1</td>
<td>i</td>
</tr>
<tr>
<td>( \chi_7 )</td>
<td>1</td>
<td>-i</td>
<td>-1</td>
<td>i</td>
<td>1</td>
<td>-i</td>
<td>1</td>
<td>-i</td>
<td>1</td>
<td>-i</td>
<td>1</td>
<td>-i</td>
<td>1</td>
<td>i</td>
</tr>
<tr>
<td>( \chi_8 )</td>
<td>1</td>
<td>i</td>
<td>-1</td>
<td>-i</td>
<td>1</td>
<td>i</td>
<td>-1</td>
<td>1</td>
<td>i</td>
<td>-1</td>
<td>1</td>
<td>i</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_9 )</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_{10} )</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_{11} )</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_{12} )</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_{13} )</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_{14} )</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Using theorem 2.15, we see that \( G_3 \) is not abelian, since six of its irreducible representation, \( \chi_9, \ldots, \chi_{14} \) has degree \( 2 \neq 1 \). Also by Lemma 2.14, we have \( ker \chi_2 = C_1 \cup C_3 \cup C_5 \cup C_7 \cup C_8 \cup C_{10} \cup C_{12} \cup C_{14} \). Thus the kernel of \( \chi_2 \) is a cyclic group of order 16. Thus \( ker \chi_2 \neq \{1\} \), and by corollary 2.13, \( G_3 \) is not simple. The entries in the character table for \( C_2, C_5, C_6 \) and \( C_{12} \) are complex conjugates of the corresponding entries of \( C_4, C_{10}, C_9 \) and \( C_{14} \).

**Illustration 3.4**

Consider the group:
\[ G_4 = \{(1,1,2),(1,1,2),(1,5,2,6),(3,7,4),(8),(1,6,2,5),(3,7,4),(8),(1,2,5,6),(3,7,4),(8),(1,6,2,5),(3,7,4),(8),(1,2,5,6),(3,7,4),(8),(1,6,2,5),(3,7,4),(8),(1,6,2,5)\} \]
Now, $|G_4| = 32 = 2^5$, which is a p-group.

The group $G_4$ has fourteen conjugacy classes and is represented by:

\[ C_1 = \{1\}, \quad C_2 = \{(7,8),(3,4)\} \quad C_3 = \{(5,6),(1,2)\} \quad C_4 = \{(5,6)(7,8),(1,2)(3,4)\} \]

\[ C_5 = \{(3,4)(7,8)\} \quad C_6 = \{(3,4)(5,6),(1,2)(7,8)\} \quad C_7 = \{(3,4)(5,6)(7,8),(1,2)(3,4)\} \]

\[ C_8 = \{(1,2)(5,6)\} \quad C_9 = \{(1,2)(5,6)(7,8),(1,2)(3,4)(5,6)\} \quad C_{10} = \{(1,2)(3,4)(5,6)(7,8)\} \]

\[ C_11 = \{(1,5)(2,6)(3,7)(4,8),(1,5)(2,6)(3,8)(4,7),(1,6)(2,5)(3,7)(4,8),(1,6)(2,5)(3,8)(4,7)\} \]

\[ C_12 = \{(1,5)(2,6)(3,7)(4,8),(1,5)(2,6)(3,8)(4,7),(1,6)(2,5)(3,7)(4,8),(1,6)(2,5)(3,8)(4,7)\} \]

\[ C_13 = \{(1,5,2,6)(3,7)(4,8),(1,5,2,6)(3,8)(4,7),(1,6,2,5)(3,7)(4,8),(1,6,2,5)(3,8)(4,7)\} \]

\[ C_{14} = \{(1,5,2,6)(3,7,4,8),(1,5,2,6)(3,8,4,7),(1,6,2,5)(3,7,4,8),(1,6,2,5)(3,8,4,7)\} \]

We observed that there is an element $g$ for which $g$ and $g^{-1}$ belong to the same conjugacy classes. i.e. In $C_{13}$ and $C_{14}$. Correspondingly by theorem 2.17 $G_5$ have fourteen irreducible representations. From the relation in corollary 2.12,

\[ |G_4| = 1^2 + 1^2 + 1^2 + 1^2 + 2^2 + 2^2 + 2^2 + 2^2 + 2^2 = 32 \]

Therefore, $G_4$ must have exactly four in equivalent one-dimensional and seven two-dimensional irreducible representations.

Since $\chi_1(1)$, gives the dimension of the representation, may therefore fill out the first column in the character table. Following the same procedure as in illustration 3.1 and theorem 2.16, we obtained the complete character table of $G_4$ as

<table>
<thead>
<tr>
<th>Character Table of $G_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Rep$ $g$</td>
</tr>
<tr>
<td>$</td>
</tr>
<tr>
<td>$\chi_1$</td>
</tr>
<tr>
<td>$\chi_2$</td>
</tr>
<tr>
<td>$\chi_3$</td>
</tr>
<tr>
<td>$\chi_4$</td>
</tr>
<tr>
<td>$\chi_5$</td>
</tr>
<tr>
<td>$\chi_6$</td>
</tr>
<tr>
<td>$\chi_7$</td>
</tr>
<tr>
<td>$\chi_8$</td>
</tr>
<tr>
<td>$\chi_9$</td>
</tr>
<tr>
<td>$\chi_{10}$</td>
</tr>
</tbody>
</table>

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Using theorem 2.15, we see that $G_5$ is not abelian, since six of its irreducible representation, $\chi_9, \ldots, \chi_{14}$ has degree $2 \neq 1$. Also by Lemma 2.14, we have $\ker \chi_2 = C_1 \cup C_4 \cup C_5 \cup C_6 \cup C_9 \cup C_{10} \cup C_{12} \cup C_{13}$. Thus the kernel of $\chi_2$ is a cyclic group of order 16. Therefore $\ker \chi_2 \neq 1$, and by corollary 2.13, $G_5$ is not simple. From the table above, we that all the entries of character table for $C_{13}$ and $C_{14}$ are real.

**Remarks**

In this paper, the following were observed:

a. If $g$ and $g^{-1}$ belongs to different conjugacy class say $C_i$ and $C_j$ then the entries in the character table for $C_i$ are complex conjugate of the corresponding entries for $C_j$.

b. If $g$ and $g^{-1}$ belongs to the same conjugacy class $C_i$ then the entries in the character table for $C_i$ are real valued.

c. If there is element $g$ for which $g^{-1}$ does not belongs to any conjugacy class, the entries in the character table are real valued.

d. Groups of order $2^n$ for $3 \leq n \leq 6$ are non abelian and not simple while group of orders $n \leq 2$ are abelian.

### 4.2 THUS, validation our results

4.2.1 Validating our results in 3.1

```
gap> G1:=Dihedralgroup(IsGroup,8);Group([ (1,2,3,4), (2,4) ]);  
gap> Order(G1);8  
gap> C1:=conjugacyclasses(G1);[(1)^g,(2,4)^g,(1,2)(3,4)^g,(1,2,3,4)^g, (1,3)(2,4)^g ]  
gap> size(C1);5  
gap> CT1:=Charactertable(G1);Charactertable(Group([ (1,2,3,4),(2,4) ]))  
gap> Isabelian(CT1);false  
gap> Issimple(CT1);false  
gap> Display(CT1,rec(powermap:=false,centralizers:=false));CT1  
```

```
1a 2a 2b 4a 2c  
X.1  1  1  1  1  
X.2  1 -1 -1  1  
X.3  1 -1  1 -1  
X.4  1  1 -1 -1  
X.5  2 . . . -2  
```

4.2.2 Our results in 3.2, validated as:

```
gap> G2:=dihedralgroup(IsGroup,16);group([ (1,2,3,4,5,6,7,8),(2,8)(3,7)(4,6) ]);  
gap> Order(G2);16  
```
 gapsize(G2);7
 CT2:=charactertable(G3);charactertable(group([1,2,3,4,5,6,7,8, 2,8,3,7,4,6]));
 Isabelian(CT2);false
 Issimple(CT2);false
 Display(CT2,rec(powermap:=false,centralizers:=false));CT2

 1a 2a 2b 8a 4a 8b 2c
 X.1 1 1 1 1 1 1 1 1
 X.2 1 -1 -1 1 1 1 1 1
 X.3 1 -1 1 -1 1 1 1 1
 X.4 1 1 -1 -1 1 1 1 1
 X.5 2 . . . -2 . 2
 X.7 2 . . -A . A -2
 A = -E(8)+E(8)^3
 = -ER(2) = -R2

 4.2.3 Thus, validating our results in 3.3

 gapsize(G3);14
 CT3:=Charactertable(G4);Charactertable(group([1,2,3,4,5,6,7,8, 1,5,2,6,3,7,4,8]));
 Isabelian(CT3);false
 Issimple(CT3);false
 Display(CT3,rec(powermap:=false,centralizers:=false));CT3

 1a 4a 2a 4b 4c 4d 4e 2b 4f 4g 2c 8a 4h 8b
 X.1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
 X.2 1 -1 1 -1 1 -1 1 -1 1 -1 1 -1 1
 X.3 1 -1 1 -1 1 -1 1 -1 1 -1 1 -1 1
 X.4 1 1 1 1 1 1 1 1 1 -1 -1 -1 -1
 X.5 1 A -1 -A A -1 A -1 -1 A 1 A
 X.6 1 -A -1 A -1 A 1 -A -1 -1 A 1 A
 X.7 1 A -1 -A A -1 A -1 -1 A 1 A
 X.8 1 -A -1 A -1 A 1 -A -1 -1 A 1 A
X.9  2 . 2 . -2 2 . -2 . . .
X.10  2 . -2 . 2 . -2 2 . 2 . . .
X.11  2  B . /B C -/B . -2 -B -C . . .
X.12  2 /B . B -C -B . -2 -/B C . . .
X.13  2 -/B . -B -C B . -2 /B C . . .
X.14  2 -B . /B C /B . -2 B -C . . .

A = -E(4)
  = -ER(-1) = -I
B = -1-E(4)
  = -1-ER(-1) = -1-I
C = 2*E(4)
  = 2*ER(-1) = 2I

4.2.4 validating our results in 3.4

```
Gap> a:=Group((1,2),(3,4));Group([ (1,2), (3,4) ])
Gap> b:=group((5,6));group([ (5,6) ])
Gap>G4:=Wreathproduct(a,b);Group([((1,2),(3,4),(5,6),(7,8),(1,5)(2,6)(3,7)(4,8))])
Gap> Order(G4);32
Gap>C4:=Conjugacyclasses(G4);[(1)^g,(7,8)^g,(5,6)^g,(5,6)(7,8)^g,(3,4)(7,8)^g,(3,4)(5,6)^g,(3,4)(5,6)(7,8)^g,(1,2)(5,6)^g,(1,2)(3,4)(5,6)(7,8)^g,(1,5)(2,6)(3,7,4,8)^g,(1,5,2,6)(3,7)(4,8)^g,(1,5,2,6)(3,7,4,8)^g]
Gap> Size(C4);14
Gap> CT5:=Charactertable(G4);Charactertable(Group([((1,2),(3,4),(5,6),(7,8),(1,5)(2,6)(3,7)(4,8))])
Gap> Isabelian(CT4);false
Gap> Issimple(CT4);false
Gap> Display(C4,rec(powermap:=false,centralizers:=false)); CT4
```

1a 2a 2b 2c 2d 2e 2f 2g 2h 2i 2j 4a 4b 4c

X.1  1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
X.2  1 -1 1 1 -1 1 -1 1 1 -1 1 1 -1 1
X.3  1 -1 1 1 -1 1 -1 1 1 -1 1 1 -1 1
X.4  1 -1 1 1 -1 1 -1 1 1 -1 1 1 -1 1
X.5  1 -1 1 1 -1 1 -1 1 1 -1 1 1 -1 1
X.6  1 1 -1 1 -1 1 1 -1 1 1 -1 1 1 -1 1
X.7  1 1 -1 1 -1 1 1 -1 1 1 -1 1 1 -1 1
X.9  2 2 . -2 . -2 2 . -2 . . .
X.10  2 2 . -2 . 2 2 . -2 . . .
X.11  2 2 . -2 . -2 2 . -2 . . .
X.12  2 2 . -2 . 2 2 . -2 . . .
X.13  2 2 . -2 . 2 2 . -2 . . .
X.14  2 2 . -2 . -2 2 . -2 . . .
REFERENCES