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Representation of p-Groups by character tables

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ABSTRACT

In this paper, we applied some group concepts to construct some p-groups as well as to display their nature, which represent them by character tables. It was observed that if g *and* g^{-1} *belongs to different conjugacy class say* C_i *and* C_j *then the entries in the character table for* C_j *are complex conjugate of the corresponding entries for* C_i*. Also, if* g *and* g^{-1} *belongs to the same conjugacy class* C_i *then the entries in the character table for* C_i *are real valued. We also applied the Groups, Algorithms and Programming (GAP) version 4.4.12 to assist towards validations of result.*

INTRODUCTION

Representation theory is concerned with the ways of writing a group as a group of matrices. Not only is the theory beautiful in its own right, but also provides one of the keys to a proper understanding of finite groups. For example, it is often vital to have a concrete description of a particular group; this is achieved by finding a representation of the group as a group of matrices.

2. Preliminaries

To begin, with there is the need for some preliminary fact and brief discussion of notations. Some of them can be verified quite readily.

2.1 Definition

Let A G_1 and G_2 be groups. A group homomorphism is a function $\phi: G_1 \to G_2$ such that for all $a, b \in G_1, \phi(ab) = \phi(a).\phi(b)$ and $f(a^{-1}) = (f(a))^{-1}$.
An invertible group homomorphism is called group isomorphism.

2.2 Definition

A representation of a group G with representation space V is a homomorphism
$\rho: g \to \rho(g)$ of G into $GL(V)$.
From the homomorphism property we have for $g, h \in G$:

$v\rho(gh) = v\rho(g)\rho(h)$,
$v\rho(1) = v1_V$

2.3 Definition
Two representations $\rho, \varphi: G \to GL(n, F)$ are said to be equivalent if there exist a n by n matrix P over F such that
$P^{-1}(g)P = \varphi(g)$ for all $g\epsilon G$

2.4 Theorem
Let $\rho(g)$ be a matrix representation of G. Then the character $\chi(g)$ of ρ has the following properties
(i) Equivalent representations have the same character.
(ii) If g and h are conjugate in G, then$\chi(g) = \chi(h)$.
Proof: (i) If $\varphi(g)$ and $\rho(g)$ are equivalent representations, the by a well known result, $\varphi(g)$ and $\rho(g)$ have the same characteristic matrix. Thus the respective coefficient of λ^{m-1} are equal ie

$$b_{11}(g) + b_{22}(g) + \cdots .. + b_{nn}(g) = a_{11}(g) + a_{22}(g) + \cdots ... + a_{nn}(g)$$

Hence equivalent representations have the same character.
(i) Suppose that g and h are conjugate elements via t in G, Then $h = t^{-1}gt$.
Thus in any matrix representation $\rho(g)$ we have
$\rho(h) = \rho(t^{-1}gt) = \rho(t^{-1})\rho(g)\rho(t)$, since ρ is a representation.
Identifying $\rho(t)$ with T in $\varphi(g) = T^{-1}\rho(g)T$ we find that $\rho(h)$ and $\rho(g)$ are equivalent representation. Hence by (i)
$tr\rho(h) = tr\rho(g)$ ie $\chi(g) = \chi(h)$.

2.5 Definition
Let $\rho: G \to GL(n, F)$ be a representation of a group G over a field F. The function $\chi: G \to$ F defined by $\chi(g) = tr\big(\rho(g)\big)$ is called character of ρ.
The character satisfies the following properties:
1. $\chi_\rho(e) = \deg(\rho)$.
2. $\chi_\rho(xgx^{-1}) = \chi_\rho(g)\ \forall\ x, g\ \in G$.
3. $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$

2.6 Particular type of characters:
(i) Trivial character: The character corresponding to the trivial representation of G is called trivial character of G. This character has value 1 over all elements of G, $(\chi(g) = 1,\ \forall\ g \in G)$.
(ii) Linear Character: A character of degree 1 is called a linear character. The first step in constructing character table of G, which carries complete description of the structure of the group G, is often writing down all the linear characters.
(iii) Permutation character: A permutation character χ_π is the character afforded by a permutation representation $\pi: G \to S_n$.

2.7 Lemma
Suppose $F = \mathbb{C}$ and let χ be character of G. Then $\chi(g^{-1}) = \overline{\chi(g)}\ \forall\ g \in G$, where $\overline{\chi(g)}$ denote the complex conjugate of $\chi(g)$.

2.8 Definition
Let χ and ψ be characters of G. Then inner product is defined as
$\langle\chi, \psi\rangle = |G|^{-1} \sum_{g\epsilon G} \chi(g)\psi(g^{-1})$

Since summing over all $g \in G$ is the same as summing over all $g^{-1} \in G$, we have $\langle \chi, \psi \rangle = \langle \psi, \chi \rangle$. Also by Lemma 2.7

$$\langle \chi, \chi \rangle = |G|^{-1} \sum_{g \epsilon G} \chi(g)\chi(g^{-1}) = |G|^{-1} \sum_{g \epsilon G} \chi(g)\overline{\chi(g)}$$
$$= |G|^{-1} \sum_{g \epsilon G} |\chi(g)|^2 > 0$$
$$\overline{\langle \chi, \psi \rangle} = |G|^{-1} \sum_{g \epsilon G} \overline{\chi(g)\psi(g^{-1})} = |G|^{-1} \sum_{g \epsilon G} \chi(g^{-1})\psi(g)$$
$$= \langle \psi, \chi \rangle = \langle \chi, \psi \rangle.$$

Hence $\langle \chi, \psi \rangle \in \mathbb{R}$.

2.9 Theorem

Let χ and ψ be characters of two non-isomorphic irreducible representations of G. Then we have

(i) $\langle \chi, \psi \rangle = 0,$

(ii) $\langle \chi, \chi \rangle = 1$

2.10 Corollary

If $\chi_1, \dots \dots \dots., \chi_k$ are all the irreducible characters of G and $\chi = \sum_{i=1}^{k} n_i \chi_i$ and
$\phi = \sum_{i=1}^{k} m_i \chi_i$
are any two characters of G, then
$\langle \chi, \phi \rangle = \sum_{i=1}^{k} n_i m_i$.

2.11 Lemma

Let χ be character of G. Then χ is irreducible if and only if $\langle \chi, \chi \rangle = 1$. This is a criterion for the irreducibility of a character.

Proof:

($\Rightarrow$) This was proved in theorem 2.9

($\Leftarrow$) If $\chi_1 \dots \dots \dots . \chi_{k.}$ are all irreducible characters of G, we can express as
$\chi = \sum_{i=1}^{k} n_i \chi_i$ $n_i \geq 0$. Assume that $\langle \chi, \chi \rangle = 1$, the by corollary 2.11
$\langle \chi, \chi \rangle = \sum_{i=1}^{k} n_i^2 = 1$. Since $n_i \in \mathbb{Z}$ and $n_i \geq 0$ we have that for one i, $n_i = 1$ and for all $j \neq i$, $n_j = 0$, and so $\chi = \chi_i$ and χ is irreducible.

For the next two lemmas we recall the character χ_R of the regular representation of G. For $F = \mathbb{C}$ and $g \in G$:

$$\chi_R(g) = \begin{cases} |G|, & if\ g = 1 \\ 0, & otherwise \end{cases}$$

2.12 Corollary

Let $r_1, r_2, \dots \dots \dots \dots, r_s$ be the irreducible representations of G. Then
$|G| = \sum n_i^2$
Where n_i is the dimension of the representation r_i.

2.13 Corollary

Let χ_1 be the character of the trivial representation. The G is simple iff
$ker\chi_i = 1\ \ for\ \ 2 \leq i \leq k.$

2.14 Lemma

If χ is a character of G, then $ker\chi = \{ g \in G \mid \chi(g) = \chi(1) \}$.

2.15 Theorem

A group G is abelian if and only if every irreducible character χ_i of G is linear.

2.16 Theorem

Let $\chi_{1,} \dots \dots \dots ., \chi_k$ be all irreducible characters of G and let $g_{1,} \dots \dots \dots ., g_k$ be the representatives of the conjugacy classes $C_1, \dots \dots \dots ., C_k$ of G.

Then we have

(i) The row orthogonality relation

$$\sum_{\alpha=1}^{k} \frac{\chi_i(g_\alpha)\overline{\chi_\iota(g_\alpha)}}{c_G(g_\alpha)} = \delta_{ij} \;\; \forall\, i, j = 1, \dots \dots . . k;$$

(ii) The column orthogonality relation

$$\sum_{\alpha=1}^{k} \chi_i(g_\alpha)\overline{\chi_J(g_\alpha)} = \delta_{\alpha\beta}|C_G(g_\alpha)| \;\; \forall\, \alpha, \beta = 1, \dots \dots ., k,$$

Where we again define $\delta_{ij} = \begin{cases} 1, & if\ i = j \\ 0, & if\ i \neq j \end{cases}$.

2.17 Theorem

The number of irreducible characters of a group G is equal to the number of conjugacy classes of G.

2.18 Lemma

If $\chi_1 \dots \dots \dots . \chi_{k.}$ are all the irreducible characters of G, then
$\sum_{i=1}^{k} \chi_i{}^2(1) = |G|$.

Proof: $|G| = \chi_R\,(1) = \left[\sum_{i=1}^{k} \chi_i(1)\chi_i\right](1) = \sum_{i=1}^{k} \chi_i{}^2(1)$.

RESULTS

Illustration 3.1

Consider the p-group G_1

G_1 = {(1), (1 3),(2 4),(12)(34),(13)(24), (1 4)(2 3), (1234),(1432) }

Now, $|G_1|$ = 8 = 2^3, which is a p-group.

The conjugacy classes of G_2 are:

C_1 = (1), C_2 = (12), (34), C_3 = (12)(34), (14)(23)

C_4 =(13)(24), C_5 = (1234), (1432).

It will be observed from the conjugacy classes of C_5 above, for any $g \in G$ g and g^{-1} belong to the same conjugacy class.

According to By theorem 2.17, , there are five irreducible representations for the group G_1. By corollary 2.12, we find a set of five positive integers, l_1, l_2, l_3, l_4 and l_5 which satisfy the equation $l_1{}^2 + l_2{}^2 + l_3{}^2 + l_4{}^2 + l_5{}^2 = 8$. The only values of $l_i\,(i = 1, \dots 5)$ which satisfy this requirement are 1, 1, 1, 1 and 2. Thus, the group G_1 has four

1-dimensional irreducible representations and one 2-dimensional irreducible representation. Set χ_1 *to* be the trivial character: $\chi_1(1) = 1\ \forall\ g \in G_1$. That is by Lemma 2.18, in any group, there will be 1-dimensional representations whose character are all equal to 1, since

$\sum_R (\chi_1(R))^2 = (1)1^2 + (2)1^2 + (1)1^2 + (2)1^2 + (2)1^2 = 8.$

The other representations will have to be such that

$\sum_R (\chi_1(R))^2 = 8$

which can be true if and only if each $\chi_i(R) = \pm 1$. By Theorem 2.16, each of the other three representations has to be orthogonal to the first irreducible representation, χ_1.

Thus, there will have to be two +1's and two -1's. The fifth representation will be of dimension 2, hence $\chi_5(C_1) = 2$. In order to find out the values of $\chi_5(C_2), \chi_5(C_3)$, $\chi_5(C_4)$ and $\chi_5(C_5)$, the orthogonality relationships in Theorem 2.17 will be used:

Partial character table of G_1:

$Rep\ g$	C_1	C_2	C_3	C_4	C_5
$\lvert C_g \rvert$	1	2	2	1	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	-1	1	1	-1
χ_4	1	1	-1	-1	1
χ_5	a	b	c	d	e

Orthogonality of χ_5 with the other known χ_i's we obtained the following set of equations.

$\langle \chi_5, \chi_1 \rangle = 2 + 2a + 2b + c + 2d = 0$ (3.1)

$\langle \chi_5, \chi_1 \rangle = 2 + 2a + 2b + c - 2d = 0$ (3.2)

$\langle \chi_5, \chi_1 \rangle = 2 + 2a + 2b + c - 2d = 0$ (3.3)

$\langle \chi_5, \chi_1 \rangle = 2 + 2a - 2b + c + 2d = 0$ (3.4)

Adding (3.1) and (3.2), (3.3) and (3.4) we got

$2 + 2a + c = 0$ (3.5)

$2 - 2a + c = 0$ (3.6)

Solving (3.5) and (3.6) we have $a = 0, c = -1$

Substituting the values of a and c into equations (3.1) and (3.2) and solving it, we have

$b = d = 0$

Complete character table of G_2:

$Rep\ g$	C_1	C_2	C_3	C_4	C_5
$\lvert C_g \rvert$	1	2	2	1	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	-1	1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	0	0	-2	0

The characters of G_1 are all irreducible since by theorem 2.12, that is

$\langle \chi_1, \chi_1 \rangle = \langle \chi_2, \chi_2 \rangle = \langle \chi_3, \chi_3 \rangle = \langle \chi_4, \chi_4 \rangle = \langle \chi_5, \chi_5 \rangle = 1$

Also by 2.19, $\sum_{i=1}^{k} \chi_i^2\ (1) = |G_1|$ and by 2.9 $\langle \chi_1, \chi_1 \rangle = 0$

With theorem 2.15, we see that G_1 is not abelian, since one of its irreducible representation, χ_5 has degree $2 \neq 1$. Also by Lemma 2.14, we have

$ker\chi_1 = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 = |G_3|$, $ker\chi_2 = C_1 \cup C_2 \cup C_4$, $ker\chi_3 = C_1 \cup C_3 \cup C_4$, $ker\chi_4 = C_1 \cup C_2 \cup C_5$ and $ker\chi_5 = C_1$.

Thus the kernel of χ_2 is a cyclic group of order 4. By corollary 2.13, $ker\chi_2 \neq 1$, and, G_1 is not simple. From the character table of G_1 above, the entries of characters of C_5 on table are real.

Illustration 3.2
Consider the group:
G_2={(1),(1,5)(2,6)(3,7)(4,8),(1,7,5,3)(2,8,6,4),(1,3,5,7)(2,4,6,8),(1,8,7,6,5,4,3,2),(1,4,7,2,5,8,3,6),(1,6,3,8,5,2,7,4),(1,2,3,4,5,6,7,8),(2,8)(3,7)(4,6),(1,5)(2,4)(6,8),(1,7)(2,6)(3,5),(1,3)(4,8)(5,7),(1,8)(2,7)(3,6)(4,5),(1,4)(2,3)(5,8)(6,7),(1,6)(2,5)(3,4)(7,8), (1,2)(3,8)(4,7)(5,6)}
Now, $|G_2|$ = 16 = 2^4 , which is a p-group.
G_2 has seven conjugacy classes, namely
C_1 =1 C_2 = {1357) (2468), (1753) (2864)} C_3 = {12345678), (18765432)}
C_4 = {14725836), (16385274)} C_5 = (15) (26) (37) (48)
C_6={12)(38)(47)(56),(14)(23)(58)(67),(16)(25)(34)(78),(18)(27)(36)(45)}
C_7 = {(13) (48)(57), (15)(24)(68),(17)(26)(35),(28)(37)(46)}

We noticed above, C_2, C_3 *and* C_4 are conjugate to their inverses.
By theorem 2.17, G_2 has seven irreducible representations.
$|G_2| = 1^2 + 1^2 + 1^2 + 1^2 + 2^2 + 2^2 + 2^2 = 16$ by corollary 2.12.
Set χ_1 *to* be the trivial character: $\chi_1(1) = 1 \;\forall\; g \in G_2$.
Set χ_2 *to* be a sign character: $\chi_2(g) = 1$ if g is even permutation, and $\chi_2(g) = -1$ if g is odd.
We will use theorem 2.16 to obtained all the linear characters of G_2. χ_5 has dimension 2, it must be orthogonal to χ_1, χ_2, χ_3 *and* χ_4 . Therefore one of the characters must be -2.

The partial character table of G_2 is:

Rep g	C_1	C_2	C_3	C_4	C_5	C_6	C_7
$\lvert C_g\rvert$	1	2	2	2	1	4	4
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1
χ_3	1	1	1	1	1	-1	1
χ_4	1	-1	1	-1	1	-1	1
χ_5	2	0	0	0	-2	0	0
χ_6	2	a_1	b_1	c_1	d_1	e_1	f_1
χ_7	2	a_2	b_2	c_2	d_2	e_2	f_2

Now by orthogonality relation theorem 2.16, we obtained the entries of the rows 6 and 7 as follows:

Using the orthogonality of the first and second columns we got
1.1+ 1.(-1) + 1.1 + 1.)-1) + 2.0 + $2a_1$+$2a_1$ $= 0$
$a_1 + a_1 = 0$ (3.13)
1.1+(-1).(-1)+1.1+(-1).(-1)+0.0+$a_1.a_2 = \frac{16}{2}$= 8
$a_1^2 + a_2^2 = 4$ (3.14)
Solving (3.13) and (3.14) we have
$a = -\sqrt{2}$ and $b = \sqrt{2}$
Similarly, the orthogonality of first and fifth columns
$d_1 + d_2 = 0$ (3.15)
$d_1^2 + d_2^2 = 8$ (3.16)
Solving (3.15) and (3.16) we have
d_1 = -2, d_2 = 2

Orthogonality of first and sixth columns

$e_1 + e_2 = 0$ (3.17)
$e_1^2 + e_2^2 = 0$ (3.18)
Solving (3.17) and (3.18) we have $e_1 = e_2 = 0$
The orthogonality of the first and seventh columns
$f_1 + f_2 = 0$ (3.19)
$f_1^2 + f_2^2 = 0$ (3.20)
Solving (3.19) and (3.20) we have, $f_1 = f_1 = 0$
Using the orthogonality of first and third columns
$2+b_1 + b_2 = 0$ (3.21)
$b_1^2 + b_2^2 = 4$ (3.22)
Solving (3.21) and (3.22) we have b_1= 0, b_2= -2
The orthogonality of first and fourth columns
$c_1 + c_2 = 0$ (3.23)
$c_1^2 + c_2^2 = 4$ (3.24)
Solving (3.23) and (3.24) we have $c_1 = \sqrt{2}$, $c_2 = -\sqrt{2}$

This complete the character table of G_2

$Rep\ g$	C_1	C_2	C_3	C_4	C_5	C_6	C_7
$\lvert C_g \rvert$	1	2	2	2	1	4	4
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1
χ_3	1	1	1	1	1	–1	1
χ_4	1	-1	1	-1	1	–1	1
χ_5	2	0	0	0	-2	0	0
χ_6	2	$-\sqrt{2}$	0	$\sqrt{2}$	-2	0	0
χ_7	2	$\sqrt{2}$	-2	$-\sqrt{2}$	2	0	0

Using theorem 2.15, we see that G_2 is not abelian, since three of its irreducible representation, χ_5, χ_6 *and* χ_7 has degree $2 \neq 1$. Also by Lemma 2.14, we have $ker\chi_2 = C_1 \cup C_3 \cup C_5 \cup C_6$. Thus the kernel of χ_2 is a cyclic group of order 8. Thus $ker\chi_2 \neq 1$, and by corollary 2.13, G_2 is not simple. We observed from character table of G_2 that the entries of the characters of C_2, C_3 *and* C_4 are real.

Illustration 3.3
Consider the group:
G_3={(1),(1,5)(2,6)(3,7)(4,8),(1,3)(2,4),(1,4,3,2),(1,2,3,4),(1,7,3,5)(2,8,4,6),(1,8,4,7,3,6,2,5),(1,6,2,7,3,8,4,5),(5,7)(6,8),(1,3)(2,4)(5,7)(6,8),(1,4,3,2)(5,7)(6,8,)(1,2,3,4)(5,7)(6,8),(1,5,3,7)(2,6,4,8),(1,7)(2,8)(3,5)(4,6),(1,8,2,5,3,6,4,7),(1,6,4,5,3,8,2,7),(5,8,7,6),(1,3)(2,4)(5,8,7,),(1,4,3,2)(5,8,7,6),(1,2,3,4)(5,8,7,6),(1,5,4,8,3,7,2,6),(1,7,2,8,3,5,4,6),(1,8,3,6)(2,5,4,7),(1,6)(2,7)(3,8)(4,5),(5,6,7,8),(1,3)(2,4)(5,6,7,8),(1,4,3,2)(5,6,7,8),(1,2,3,4)(5,6,7,8),(1,5,2,6,3,7,4,8),(1,7,4,6,3,5,2,8),(1,8)(2,5)(3,6)(4,7),(1,6,3,8)(2,7,4,5)}.
$|G_3|$ = 32 = 2^5, which is a p-group.
The group G_3 has fourteen conjugacy classes and is represented by:
$C_1 = (1)$, C_2 ={(1,2,3,4), (5,6,7,8) } C_3 = { ((1,3)(2,4),(5,7)(6,8) },
C_4 ={ (1,4,3,2),(5,8,7,6) } C_5 ={ (1,2,3, 4)(5,6,7,8) }
C_6 ={(1,2,3, 4)(5,7)(6, 8), (1, 3),(2,4)(5,6,7,8)} C_7 ={(1,2,3,4)(5,8,7,6), (1,4,3,2)(5,6,7,8)}
C_8 ={ (1,3)(2,4)(5,7)(6,8)} C_9={(1,3)(2,4)(5,8,7,6), (1,4,3,2)(5,7)(6,8)}
C_{10}={(1,4,3,2)(5,8,7,6)}
C_{11} =(1,5)(2,6)(3,7)(4,8),(1,6)(2,7)(3,8)(4,5),(1,7)(2,8)(3,5)(4,6), (1,8)(2,5)(3,6)(4,7)}

C_{12} ={(1,5,2,6,3,7,4,8),(1,6,2,7,3,8,4,5),(1,7,2,8,3,5,4,6),(1,8,2,5,3,6,4,7)}
C_{13} = {(1,5,3,7)(2,6,4,8), (1,6,3,8)(2,7,4,5),(1,7,3,5)(2,8,4,6), (1,8,3,6)(2,5,4,7)}
C_{14} = {(1,5,4,8,3,7,2,6), (1,6,4,5,3,8,2,7), (1,7,4,6,3,5,2,8), (1,8,4,7,3,6,2,5)}

The conjugacy classes of G_3, above show that that there is an element g for which g *and* g^{-1} belong to the different conjugacy classes.
$(C_2, C_4), (C_5, C_{10}), (C_6, C_9)$ *and* (C_{12}, C_{14}).
Correspondingly by theorem 2.17 G_3 have fourteen irreducible representations. From the relation in corollary 2.12,

$$|G_3| = 1^2+1^2+1^2+1^2+1^2+1^2+1^2+1^2+2^2+2^2+2^2+2^2+2^2+2^2 = 32$$

Therefore, G_3 must have exactly eight in equivalent one-dimensional and six two-dimensional irreducible representations.
Since $\chi_1(1)$, gives the dimension of the representation, may therefore fill out the first column in the character table. Following the same procedure as in illustration 3.1 and theorem 2.16, we obtained the complete character table of G_3 as:

Character table of G_3:

Rep g	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}
$\|G_5\|$	1	2	2	2	1	2	2	1	2	1	4	4	4	4
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	–1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1
χ_3	1	–1	1	–1	1	–1	1	1	-1	1	1	-1	1	-1
χ_4	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
χ_5	1	-i	-1	i	–1	i	1	1	-i	-1	-1	i	1	i
χ_6	1	i	-1	-i	-1	-i	1	1	i	-1	-1	-i	1	i
χ_7	1	-i	-1	i	-1	i	1	1	-i	-1	1	-i	1	i
χ_8	1	i	-1	-i	-1	-i	1	1	i	-1	1	i	-1	-i
χ_9	2	0	2	0	-2	0	-2	2	0	-2	0	0	0	0
χ_{10}	2	0	-2	0	2	0	-2	2	2	0	0	0	0	0
χ_{11}	2	-1+i	0	-1-i	-2i	1+i	0	-2	1-i	2i	0	0	0	0
χ_{12}	2	-1-i	0	-1+i	2i	1-i	0	-2	-1-i	-2i	0	0	0	0
χ_{13}	2	1-i	0	1+i	-2i	-1-i	0	-2	1+i	2i	0	0	0	0
χ_{14}	2	1+i	0	1-i	-2i	-1+i	0	-2	-1-i	-2i	0	0	0	0

Using theorem 2.15, we see that G_3 is not abelian, since six of its irreducible representation, $\chi_9, \ldots\ldots, \chi_{14}$ has degree $2 \neq 1$. Also by Lemma 2.14, we have $ker\chi_2 = C_1 \cup C_3 \cup C_5 \cup C_7 \cup C_8 \cup C_{10} \cup C_{12} \cup C_{14}$. Thus the kernel of χ_2 is a cyclic group of order 16. Thus $ker\chi_2 \neq 1$, and by corollary 2.13, G_3 is not simple. The entries in the character table for C_2, C_5, C_6 *and* C_{12} are complex conjugates of the corresponding entries of C_4, C_{10}, C_9 *and* C_{14}.

Illustration 3.4
Consider the group:
G_4={(1),(1,2),(1,5)(2,6)(3,7)(4,8),(1,6,2,5)(3,7)(4,8),(5,6),(1,2)(5,6),(1,5,2,6)(3,7)(4,8),(1,6)(2,5)(3,7)(4,8),(3,4),(1,2)(3,4),(1,5)(2,6)(3,8,4,7),(1,6,2,5)(3,8,4,7),(3,4)(5,6),(1,2)(3,4)(5,6),(1,5,2,6)(3

,8,4,7),(1,6)(2,5)(3,8,4,7),(7,8),(1,2)(7,8),(1,5)(2,6)(3,7,4,8),(1,6,2,5)(3,7,4,8),(5,6)(7,8),(1,2)(5,6)(7,8),(1,5,2,6)(3,7,4,8),(1,6)(2,5)(3,7,4,8),(3,4)(7,8),(1,2)(3,4)(7,8),(1,5)(2,6)(3,8)(4,7),(1,6,2,5)(3,8)(4,7),(3,4)(5,6)(7,8),(1,2)(3,4)(5,6)(7,8),(1,5,2,6)(3,8)(4,7),(1,6)(2,5)(3,8)(4,7)}

Now, $|G_4|$ = 32 = 2^5 , which is a p-group.
The group G_4 has fourteen conjugacy classes and is represented by:
$C_1 = (1)$, C_2 ={(7,8),(3,4) } C_3 = { (5,6),(1,2) }, C_4 ={ (5,6)(7,8),(1,2)(3,4) }
C_5 ={ (3,4)(7,8)} C_6 ={(3,4)(5,6),(1,2)(7,8)} C_7 ={(3,4)(5,6)(7,8),(1,2)(3,4)(7,8)}
C_8 ={ (1,2)(5,6)} C_9={(1,2)(5,6)(7,8),(1,2)(3,4)(5,6)} C_{10}={(1,2)(3,4)(5,6)(7,8)}
C_{11} ={(1,5)(2,6)(3,7)(4,8),(1,5)(2,6)(3,8)(4,7),(1,6)(2,5)(3,7)(4,8),(1,6)(2,5)(3,8)(4,7)}
C_{12} ={(1,5)(2,6)(3,7,4,8),(1,5)(2,6)(3,8,4,7),(1,6)(2,5)(3,7,4,8),(1,6)(2,5)(3,8,4,7}
C_{13} ={(1,5,2,6)(3,7)(4,8),(1,5,2,6)(3,8)(4,7),(1,6,2,5)(3,7)(4,8),(1,6,2,5)(3,8)(4,7}
C_{14} = {(1,5,2,6)(3,7,4,8),(1,5,2,6)(3,8,4,7),(1,6,2,5)(3,7,4,8),(1,6,2,5)(3,8,4,7)}

We observed that there is an element g for which g *and* g^{-1} belong to the same conjugacy classes. i.e. In C_{13} *and* C_{14}. Correspondingly by theorem 2.17 G_5 have fourteen irreducible representations. From the relation in corollary 2.12,

$$|G_4| = 1^2 + 1^2 + 1^2 + 1^2 + 2^2 + 2^2 + 2^2 + 2^2 + 2^2 + 2^2 = 32$$

Therefore, G_4 must have exactly four in equivalent one-dimensional and seven two-dimensional irreducible representations.

Since $\chi_1(1)$, gives the dimension of the representation, may therefore fill out the first column in the character table. Following the same procedure as in illustration 3.1 and theorem 2.16, we obtained the complete character table of G_4 as

Character Table of G_4

Rep g	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}
$\lvert G_5 \rvert$	1	2	2	2	1	2	2	1	2	1	4	4	4	4
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	–1	-1	1	1	1	-1	1	-1	1	-1	1	1	-1
χ_3	1	–1	-1	1	1	1	-1	1	-1	1	1	-1	-1	1
χ_4	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1
χ_5	1	-1	1	-1	1	-1	1	1	-1	1	1	-1	1	-1
χ_6	1	1	-1	-1	1	-1	-1	1	1	1	-1	-1	1	1
χ_7	1	1	-1	-1	1	-1	-1	1	1	1	1	1	-1	-1
χ_8	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
χ_9	2	0	2	0	-2	0	-2	2	0	-2	0	0	0	0
χ_{10}	2		-2	0	-2		2	2	0	-2	0	0	0	0

		0				0								
χ_{11}	2			0	2			-2	-2	-2	0	0	0	0
		2	0			0	0							
χ_{12}	2	-		0	2			-2	2	-2	0	0	0	0
		2	0			0	0							
χ_{13}	2			2	-2	-2		-2	0	2	0	0	0	0
		0	0				0							
χ_{14}	2			-2	-2	2		-2	0	2	0	0	0	0
		0	0				0							

Using theorem 2.15, we see that G_5 is not abelian, since six of its irreducible representation, $\chi_9, \ldots\ldots, \chi_{14}$ has degree $2 \neq 1$. Also by Lemma 2.14, we have $ker\chi_2 = C_1 \cup C_4 \cup C_5 \cup C_6 \cup C_8 \cup C_{10} \cup C_{12} \cup C_{13}$. Thus the kernel of χ_2 is a cyclic group of order 16. Therefore $ker\chi_2 \neq 1$, and by corollary 2.13, G_5 is not simple. From the table above, we that all the entries of character table for C_{13} and C_{14} are real.

Remarks

In this paper, the following were observed:

a. If g and g^{-1} belongs to different conjugacy class say C_i and C_j then the entries in the character table for C_j are complex conjugate of the corresponding entries for C_i.

b. If g and g^{-1} belongs to the same conjugacy class C_i then the entries in the character table for C_i are real valued.

c. If there is element g for which g^{-1} does not belongs to any conjugacy class, the entries in the character table are real valued.

d. Groups of order 2^n for $3 \leq n \leq 6$ are non abelian and not simple while group of orders $n \leq 2$ are abelian.

4.2 THUS, validation our results

4.2.1 Validating our results in 3.1

```
gap> G1:=Dihedralgroup(Isgroup,8);Group([ (1,2,3,4), (2,4) ])
Gap> Order(G1);8
Gap>C1:=conjugacyclasses(G1);[(1)^g,(2,4)^g,(1,2)(3,4)^g,(1,2,3,4)^g, (1,3)(2,4)^g ]
Gap> size(C1);5
Gap>CT1:=Charactertable(G1);Charactertable(Group([(1,2,3,4),(2,4)]))
Gap> Isabelian(CT1);false
Gap> Issimple(CT1);false
Gap> Display(CT1,rec(powermap:=false,centralizers:=false));CT1

1a 2a 2b 4a 2c

X.1    1  1  1  1  1
X.2    1 -1 -1  1  1
X.3    1 -1  1 -1  1
X.4    1  1 -1 -1  1
X.5    2  .  .  . -2
```

4.2.2 Our results in 3.2, validated as:

```
Gap>G2:=dihedralgroup(Isgroup,16);group([(1,2,3,4,5,6,7,8),(2,8)(3,7)(4,6)])
Gap> Order(G2);16
```

Gap>
C2:=conjugacyclasses(G2);[(1)^g,(2,8)(3,7)(4,6)^g,(1,2)(3,8)(4,7)(5,6)^g,(1,2,3,4,5,6,7,8)^g,(1,3,5,7)(2,4,6,8)^g,(1,4,7,2,5,8,3,6)^g,(1,5)(2,6)(3,7)(4,8)^g]

Gap> size(C2);7
Gap>CT2:=charactertable(G3);charactertable(group([(1,2,3,4,5,6,7,8), (2,8)(3,7)(4,6)]))
Gap> Isabelian(CT2);false
Gap> Issimple(CT2);false
Gap> Display(CT2,rec(powermap:=false,centralizers:=false));CT2

	1a	2a	2b	8a	4a	8b	2c
X.1	1	1	1	1	1	1	1
X.2	1	-1	-1	1	1	1	1
X.3	1	-1	1	-1	1	-1	1
X.4	1	1	-1	-1	1	-1	1
X.5	2	.	.	.	-2	.	2
X.6	2	.	.	A	.	-A	-2
X.7	2	.	.	-A	.	A	-2

A = -E(8)+E(8)^3
= -ER(2) = -R2

4.2.3 Thus, validating our results in 3.3

Gap> a:=Group((1,2,3,4));group([(1,2,3,4)])
Gap> b:=Group((5,6));group([(5,6)])
Gap>G3:=Wreathproduct(a,b);Group([(1,2,3,4),(5,6,7,8),(1,5)(2,6)(3,7)(4,8)])
Gap> Order(G3);32
Gap>C3:=conjugacyclasses(G3);[(1)^g,(5,6,7,8)^g,(5,7)(6,8)^g,
(5,8,7,6)^g,(1,2,3,4)(5,6,7,8)^g,(1,2,3,4)(5,7)(6,8)^g,(1,2,3,4)(5,8,7,6)^g,(1,3)(2,4)(5,7)(6,8)^g,(1,3)(2,4)(5,8,7,6)^g,(1,4,3,2)(5,8,7,6)^g,(1,5)(2,6)(3,7)(4,8)^g,(1,5,2,6,3,7,4,8)^g,(1,5,3,7)(2,6,4,8)^g,(1,5,4,8,3,7,2,6)^g]

Gap> size(C3);14
Gap>CT3:=Charactertable(G4);Charactertable(Group([(1,2,3,4),(5,6,7,8),(1,5)(2,6)(3,7)(4,8)]))
Gap> Isabelian(CT3);false
Gap> Issimple(CT3);false
Gap> Display(CT3,rec(powermap:=false,centralizers:=false));CT3

	1a	4a	2a	4b	4c	4d	4e	2b	4f	4g	2c	8a	4h	8b
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1
X.3	1	-1	1	-1	1	-1	1	1	-1	1	1	-1	1	-1
X.4	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
X.5	1	A	-1	-A	-1	-A	1	1	A	-1	-1	-A	1	A
X.6	1	-A	-1	A	-1	A	1	1	-A	-1	-1	A	1	-A
X.7	1	A	-1	-A	-1	-A	1	1	A	-1	1	A	-1	-A
X.8	1	-A	-1	A	-1	A	1	1	-A	-1	1	-A	-1	A

```
X.9     2  . 2  . -2  . -2  2  . -2 . . . .
X.10    2  . -2  . 2  . -2 2  . 2 . . . .
X.11    2  B . /B  C -/B  . -2  -B -C . . . .
X.12    2 /B .  B -C  -B  . -2 -/B  C . . . .
X.13    2 -/B . -B -C  B  . -2 /B  C . . . .
X.14    2 -B . -/B  C /B  . -2  B -C . . . .
```

A = -E(4)
 = -ER(-1) = -I
B = -1-E(4)
 = -1-ER(-1) = -1-I
C = 2*E(4)
 = 2*ER(-1) = 2I

4.2.4 validating our results in 3.4

```
Gap> a:=Group((1,2),(3,4));Group([ (1,2), (3,4) ])
Gap> b:=group((5,6));group([ (5,6) ])
Gap>G4:=Wreathproduct(a,b);Group([(1,2),(3,4),(5,6),(7,8),(1,5)(2,6)(3,7)(4,8)])
Gap> Order(G4);32
Gap>C4:=Conjugacyclasses(G4);[(1)^g,(7,8)^g,(5,6)^g,(5,6)(7,8)^g,(3,4)(7,8)^g,(3,4)(5,6)^g,(3,4)(5,6)(7,8)^g,(1,2)(5,6)^g,(1,2)(5,6)(7,8)^g,(1,2)(3,4)(5,6)(7,8)^g,(1,5)(2,6)(3,7)(4,8)^g,(1,5)(2,6)(3,7,4,8)^g,(1,5,2,6)(3,7)(4,8)^g,(1,5,2,6)(3,7,4,8)^g ]
Gap> Size(C4);14
Gap>CT5:=Charactertable(G4);Charactertable(Group([(1,2),(3,4),(5,6),(7,8),(1,5)(2,6)(3,7)(4,8)]
))
Gap> Isabelian(CT4);false
Gap> Issimple(CT4);false
Gap> Display(C4,rec(powermap:=false,centralizers:=false)); CT4
```

```
1a 2a 2b 2c 2d 2e 2f 2g 2h 2i 2j 4a 4b 4c

X.1     1  1  1  1  1  1  1  1  1  1  1  1  1  1
X.2     1 -1 -1  1  1  1 -1  1 -1  1 -1  1  1 -1
X.3     1 -1 -1  1  1  1 -1  1 -1  1  1 -1 -1  1
X.4     1 -1  1 -1  1 -1  1  1 -1  1 -1  1 -1  1
X.5     1 -1  1 -1  1 -1  1  1 -1  1  1 -1  1 -1
X.6     1  1 -1 -1  1 -1 -1  1  1  1 -1 -1  1  1
X.7     1  1 -1 -1  1 -1 -1  1  1  1  1  1 -1 -1
X.8     1  1  1  1  1  1  1  1  1  1 -1 -1 -1 -1
X.9     2  .  2  . -2  . -2  2  . -2  .  .  .  .
X.10    2  . -2  . -2  .  2  2  . -2  .  .  .  .
X.11    2 -2  .  .  2  .  . -2  2 -2  .  .  .  .
X.12    2  .  . -2 -2  2  . -2  .  2  .  .  .  .
X.13    2  .  .  2 -2 -2  . -2  .  2  .  .  .  .
X.14    2  2  .  .  2  .  . -2 -2 -2  .  .  .  .
```

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