Available online at www.scholarsresearchlibrary.com

Scholars Research Library

European Journal of Applied Engineering and Scientific Research, 2014, 3 (1):21-30
(http://scholarsresearchlibrary.com/archive.html)

ISSN: 2278 – 0041

Some curious results involving certain polynomials

Salahuddin and R.K. Khola

Mewar University, Gangrar, Chittorgarh (Rajasthan) , India

ABSTRACT

In this paper we have developed some curious results involving certain polynomials.

Key Words : Hypergeometric function, Laplace Transform, Lucas Polynomials, Gegenbaur polynomials, Harmonic number, Bernoulli Polynomial, Hermite Polynomials.
2010 MSC NO: 11B39, 11B68, 33C05, 33C45, 33D50, 33D60

INTRODUCTION

We have the generalized Gaussian hypergeometric function of one variable

$$ {}_AF_B(a_1,a_2,\ldots,a_A;b_1,b_2,\ldots b_B;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \ldots (a_A)_k\, z^k}{(b_1)_k(b_2)_k \ldots (b_B)_k\, k!} \tag{1.1}$$

where the parameters $b_1, b_2, \ldots, b_B$ are neither zero nor negative integers and A , B are non negative integers.
The series converges for all finite z if $A \le B$, converges for $|z|<1$ if $A=B+1$, diverges for all z, $z \neq 0$ if $A > B+1$.

Laplace Transform
The Laplace Transform of a function *f(t)* , defined for all real numbers $t \ge 0$, is the function *F(s)*, defined by :

$$F(s) = L\{f(t)\}(s) = \int_0^{\infty} e^{-st} f(t)\,dt \tag{1.2}$$

The parameter *s* is a complex number:

$s = \sigma + i\omega$, with real number σ and ω.

Jacobi Polynomials
In mathematics, **Jacobi polynomials** (occasionally called **hypergeometric polynomials**) are a class of classical orthogonal polynomials. They are orthogonal with respect to the weight

$$(1-x)^{\alpha}(1+x)^{\beta}$$

on the interval [-1, 1]. The Gegenbauer polynomials, and thus also the Legendre, Zernike and Chebyshev polynomials, are special cases of the Jacobi polynomials.

The Jacobi polynomials were introduced by Carl Gustav Jacob Jacobi.

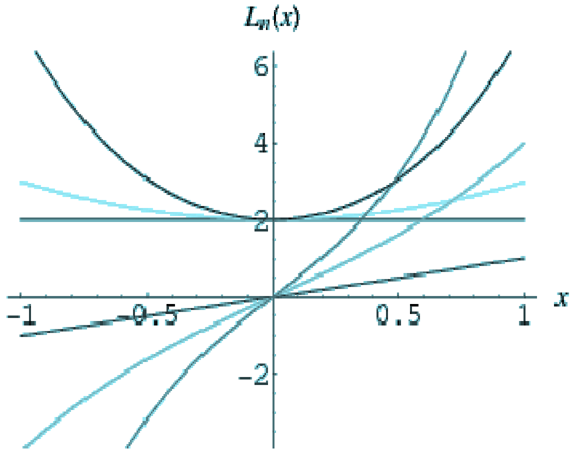
The Jacobi polynomials are defined via the hypergeometric function as follows:

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-z}{2}), \tag{1.3}$$

Where $(\alpha+1)_n$ is Pochhammer's symbol. In this case, the series for the hypergeometric function is finite, therefore one obtains the following equivalent expression:

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)} (\frac{z-1}{2})^m \tag{1.4}$$

Lucas Polynomials



The Lucas polynomials are the w-polynomials obtained by setting p(x) = x and q(x) = 1 in the Lucas polynomials sequence. It is given explicitly by

$$L_n(x) = 2^{-n}[(x-\sqrt{x^2+4}\ )^n + (x+\sqrt{x^2+4}\ )^n\,] \tag{1.5}$$

The first few are

$$\left.\begin{aligned} L_1(x) &= x \\ L_2(x) &= x^2+2 \\ L_3(x) &= x^3+3x \\ L_4(x) &= x^4+4x^2+2 \end{aligned}\right\} \tag{1.6}$$

Generalized Harmonic Number

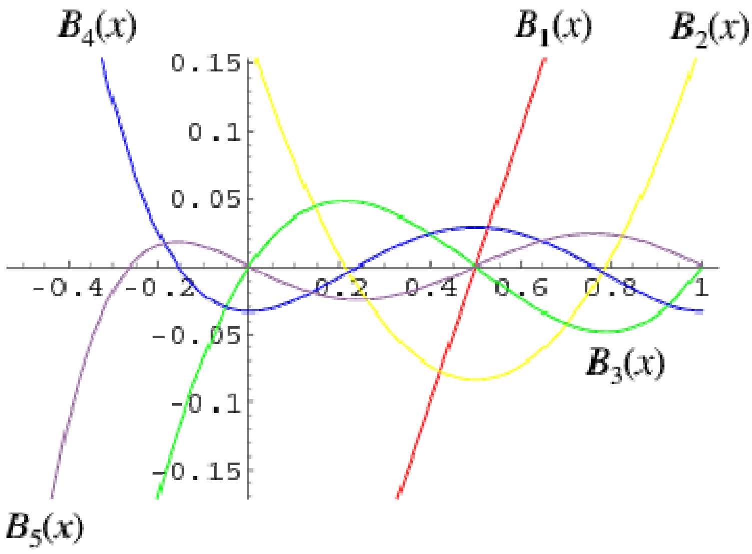
The generalized harmonic number of order *n* of *m* is given by

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m} \tag{1.7}$$

In the limit of $n \to \infty$ the generalized harmonic number converges to the Riemann zeta function

$$\lim_{n\to\infty} H_n^{(m)} = \varsigma(m) \tag{1.8}$$

Bernoulli Polynomial



In mathematics, the Bernoulli polynomials occur in the study of many special functions and in particular the Riemann zeta function and Hurwitz zeta function. This is in large part because they are an Appell sequence, i.e. a Sheffer sequence for the ordinary derivative operator. Unlike orthogonal polynomials, the Bernoulli polynomials are remarkable in that the number of crossing of the *x*-axis in the unit interval does not go up as the degree of the polynomials goes up. In the limit of large degree, the Bernoulli polynomials, appropriately scaled, approach the sine and cosine functions.

Explicit formula of Bernoulli polynomials is

$B_n(x)=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}b_{n-k}x^k$, for $n\geq 0$, where b_k are the Bernoulli numbers.

The generating function for the Bernoulli polynomials is

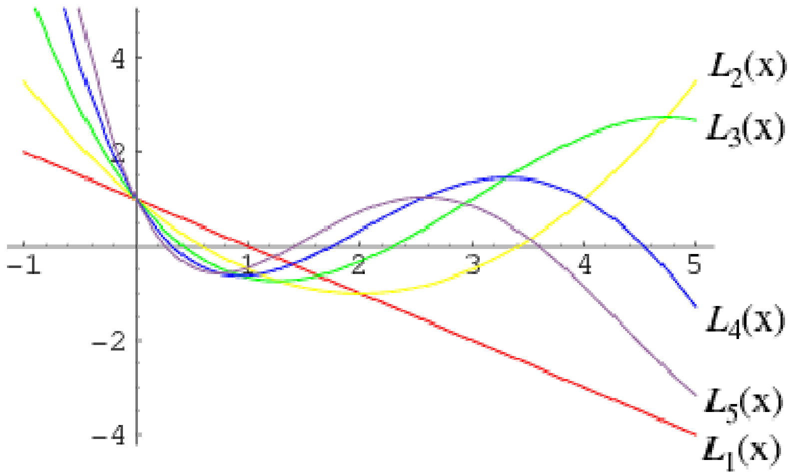
$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \tag{1.9}$$

Gegenbauer polynomials

In Mathematics, Gegenbauer polynomials or ultraspherical polynomials $C_n^{(\alpha)}(x)$ are orthogonal polynomials on the interval [-1,1] with respect to the weight function $(1-x^2)^{\alpha-\frac{1}{2}}$. They generalize Legendre polynomials and Chebyshev polynomials, and are special cases of Jacobi polynomials. They are named after Leopold Gegenbauer. Explicitly ,

$$C_n^{(\alpha)}(x)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^k\frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)k!(n-2k)!}(2z)^{(n-2k)} \tag{1.10}$$

Laguerre polynomials



The Laguerre polynomials are solutions $L_n(x)$ to the Laguerre differential equation

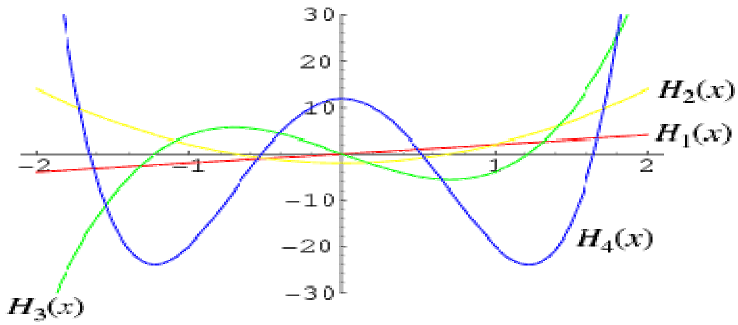
$xy''+(1-x)y'+\lambda y=0$, which is a special case of the more general associated Laguerre differential equation, defined by

$xy''+(\nu+1-x)y'+\lambda y=0$, where λ and ν are real numbers with ν=0.

The Laguerre polynomials are given by the sum

$$L_n(x)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\frac{n!}{k!(n-k)!}x^k \tag{1.11}$$

Hermite polynomials



The Hermite polynomials $H_n(x)$ are set of orthogonal polynomials over the domain (-∞,∞) with weighting function e^{-x^2}.

The Hermite polynomials $H_n(x)$ can be defined by the contour integral

$$H_n(z)=\frac{n!}{2\pi i}\oint e^{-r^2+2rz}t^{-n-1}dt,$$

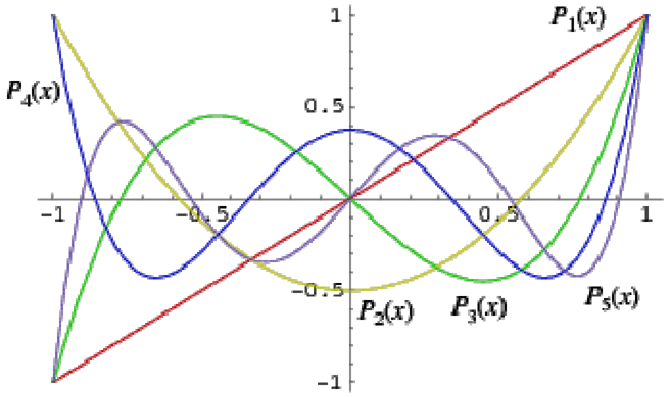
Where the contour incloses the origin and is traversed in a counterclockwise direction

(Arfken 1985, p. 416).

The first few Hermite polynomials are

$$\left.\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x \\ H_4(x) &= 16x^4 - 48x^2 + 12 \end{aligned}\right\} \qquad (1.12)$$

Legendre function of the first kind



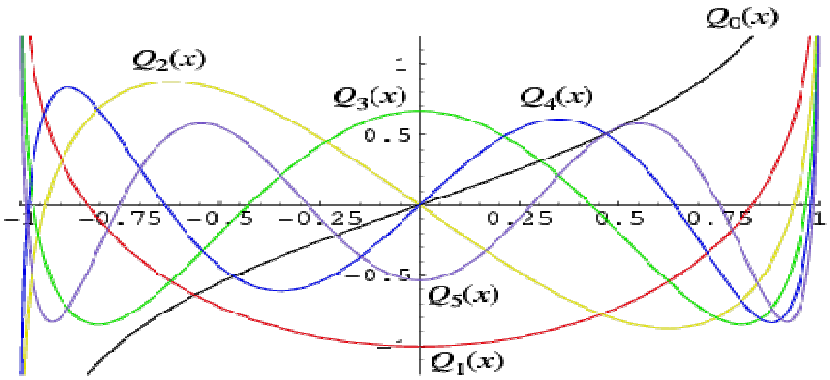
The Legendre polynomials, sometimes called Legendre functions of the first kind, Legendre coefficients, or zonal harmonics (Whittaker and Watson 1990, p. 302), are solutions to the Legendre differential equation. If *l* is an integer, they are polynomials. The Legendre polynomials $P_n(x)$ are illustrated above for $x \in [-1, 1]$ and *n*=1, 2, ..., 5.

The Legendre polynomials $P_n(x)$ can be defined by the contour integral

$$P_n(z) = \frac{1}{2\pi i} \oint (1 - 2tz + t^2)^{-\frac{1}{2}} t^{-n-1} dt \,, \qquad (1.13)$$

where the contour encloses the origin and is traversed in a counterclockwise direction (Arfken 1985, p. 416).

Legendre function of the second kind

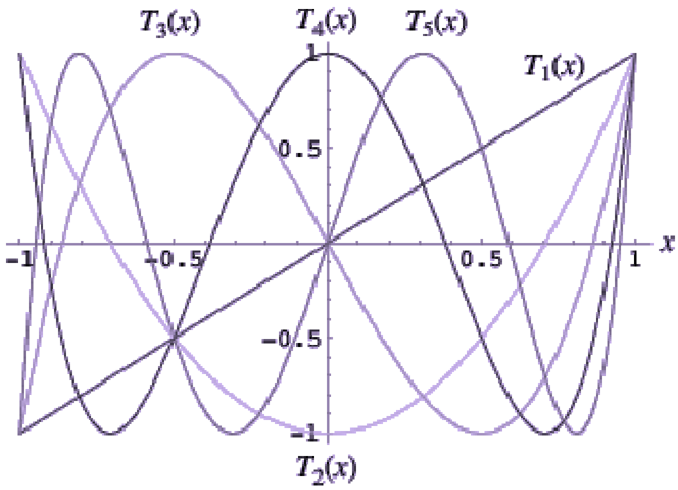


The second solution $Q_l(x)$ to the Legendre differential equation. The Legendre functions of the second kind satisfy the same recurrence relation as the Legendre polynomials.

The first few are

$$\left.\begin{aligned} Q_0(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \\ Q_1(x) &= \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)-1 \\ Q_2(x) &= \frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right)-\frac{3x}{2} \end{aligned}\right\} \tag{1.14}$$

Chebyshev polynomial of the first kind



The Chebyshev polynomials of the first kind are a set of orthogonal polynomials defined as the solutions to the Chebyshev differential equation and denoted $T_n(x)$. They are used as an approximation to a least squares fit, and are a special case of the Gegenbauer polynomial with $\alpha = 0$. They are also intimately connected with trigonometric multiple-angle formulas.

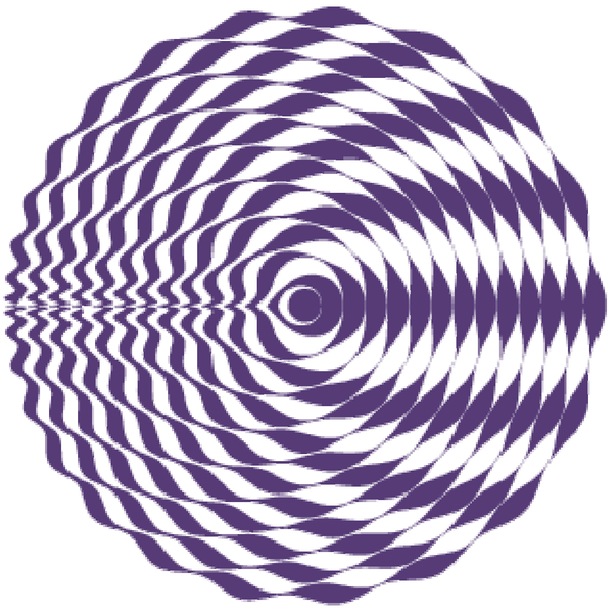
The Chebyshev polynomial of the first kind $T_n(z)$ can be defined by the contour integral

$$T_n(z) = \frac{1}{4\pi i}\oint \frac{(1-t^2)t^{-n-1}}{1-2tz+t^2}dt \, , \tag{1.15}$$

where the contour encloses the origin and is traversed in a counterclockwise direction (Arfken 1985, p. 416).
The first few Chebyshev polynomials of the first kind are

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$

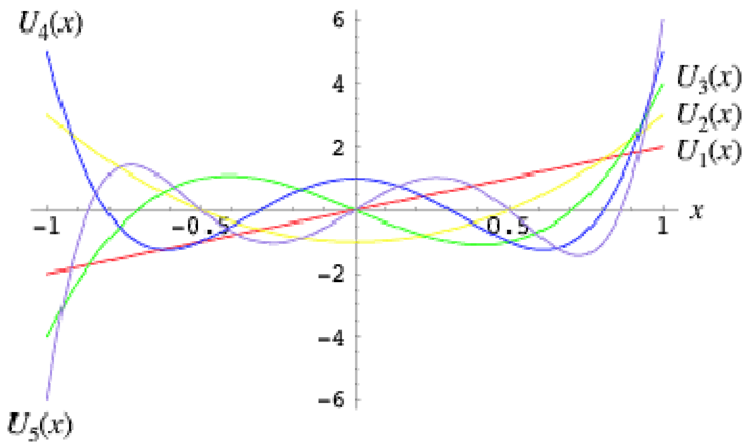
A beautiful plot can be obtained by plotting $T_n(x)$ radially, increasing the radius for each value of n , and filling in the areas between the curves (Trott 1999, pp. 10 and 84).



The Chebyshev polynomials of the first kind are defined through the identity

$T_n(\cos\theta)=\cos n\theta$.

Chebyshev polynomial of the second kind

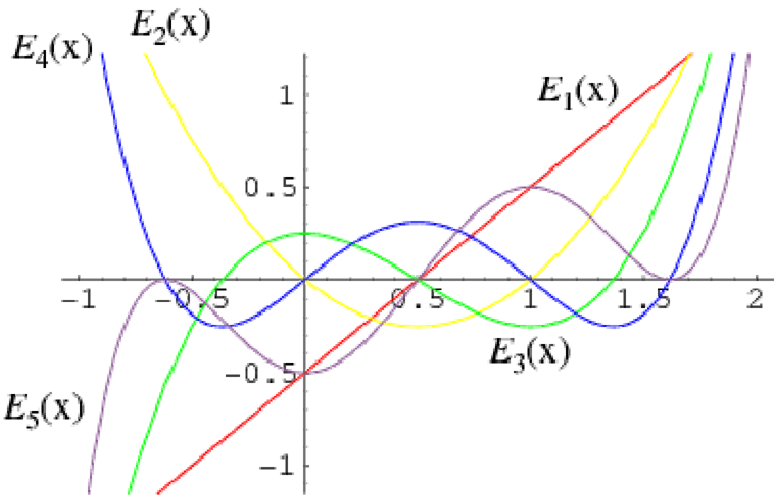


A modified set of Chebyshev polynomials defined by a slightly different generating function. They arise in the development of four-dimensionalspherical harmonics in angular momentum theory. They are a special case of the Gegenbauer polynomial with $\alpha = 1$. They are also intimately connected with trigonometric multiple-angle formulas.

The first few Chebyshev polynomials of the second kind are

$$\left.\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_2(x) &= 4x^2 - 1 \\ U_3(x) &= 8x^3 - 4x \end{aligned}\right\} \qquad (1.16)$$

Euler polynomial



The Euler polynomial $E_n(x)$ is given by the Appell sequence with

$$g(t)=\frac{1}{2}(e^t+1)\ ,$$

giving the generating function

$$\frac{2e^{xt}}{e^t+1}\equiv\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \qquad (1.17)$$

The first few Euler polynomials are

$$\left.\begin{aligned}
E_0(x)&=1\\
E_1(x)&=x-\frac{1}{2}\\
E_2(x)&=x^2-x\\
E_3(x)&=x^3-\frac{3}{2}x^2+\frac{1}{4}
\end{aligned}\right\} \qquad (1.18)$$

1. MAIN RESULTS

$$\int L_1(x)\sqrt{1+x^{-n}}\ dx=\frac{x^2(n\,{}_2F_1\left(\frac{1}{2},-\frac{2}{n};\frac{n-2}{n};-x^{-n}\right)-4\sqrt{x^{-n}+1}}{2(n-4)}+C \qquad (2.1)$$

$$\int H_1^{(x)}\sqrt{1+x^{-n}}\ dx=\frac{nx\,{}_2F_1\left(\frac{1}{2},-\frac{1}{n};\frac{n-1}{n};-x^{-n}\right)-2x\sqrt{x^{-n}+1}}{(n-2)}+C \qquad (2.2)$$

$$\int B_1(x)\sqrt{1+x^{-n}}\ dx=\frac{1}{2(n-4)(n-2)}x[\,(n-2)nx\,{}_2F_1\left(\frac{1}{2},-\frac{2}{n};\frac{n-2}{n};-x^{-n}\right)$$

$$-(n-4)n\,{}_2F_1\left(\frac{1}{2},-\frac{1}{n};\frac{n-1}{n};-x^{-n}\right)+2\sqrt{x^{-n}+1}\;(-2nx+n+4x-4)]+C \tag{2.3}$$

$$\int C_1(x)\sqrt{1+x^{-n}}\;dx=\frac{x^2\left(n\,{}_2F_1\left(\frac{1}{2},-\frac{2}{n};\frac{n-2}{n};-x^{-n}\right)-4\sqrt{x^{-n}+1}\right.}{(n-4)}+C \tag{2.4}$$

$$\int F_1(x)\sqrt{1+x^{-n}}\;dx=\frac{x^2\left(n\,{}_2F_1\left(\frac{1}{2},-\frac{2}{n};\frac{n-2}{n};-x^{-n}\right)-4\sqrt{x^{-n}+1}\right.}{2(n-4)}+C \tag{2.5}$$

$$\int L_{(x)}[1](s)\sqrt{1+x^{-n}}\;dx=\frac{nx\,{}_2F_1\left(\frac{1}{2},-\frac{1}{n};\frac{n-1}{n};-x^{-n}\right)-2x\sqrt{x^{-n}+1}}{(n-2)s}+C \tag{2.6}$$

$$\int P_1(x)\sqrt{1+x^{-n}}\;dx=\frac{x^2\left(n\,{}_2F_1\left(\frac{1}{2},-\frac{2}{n};\frac{n-2}{n};-x^{-n}\right)-4\sqrt{x^{-n}+1}\right.}{2(n-4)}+C \tag{2.7}$$

$$\int T_1(x)\sqrt{1+x^{-n}}\;dx=\frac{x^2\left(n\,{}_2F_1\left(\frac{1}{2},-\frac{2}{n};\frac{n-2}{n};-x^{-n}\right)-4\sqrt{x^{-n}+1}\right.}{2(n-4)}+C \tag{2.8}$$

$$\int U_1(x)\sqrt{1+x^{-n}}\;dx=\frac{x^2\left(n\,{}_2F_1\left(\frac{1}{2},-\frac{2}{n};\frac{n-2}{n};-x^{-n}\right)-4\sqrt{x^{-n}+1}\right.}{(n-4)}+C \tag{2.9}$$

$$\int H_1(x)\sqrt{1+x^{-n}}\;dx=\frac{x^2\left(n\,{}_2F_1\left(\frac{1}{2},-\frac{2}{n};\frac{n-2}{n};-x^{-n}\right)-4\sqrt{x^{-n}+1}\right.}{(n-4)}+C \tag{2.10}$$

$$\int L_1(x)\sqrt{1+x^{-n}}\;dx=-\frac{1}{2(n-4)(n-2)}x[(n-2)nx\,{}_2F_1\left(\frac{1}{2},-\frac{2}{n};\frac{n-2}{n};-x^{-n}\right)$$

$$-2(n-4)n\,{}_2F_1\left(\frac{1}{2},-\frac{1}{n};\frac{n-1}{n};-x^{-n}\right)+4\sqrt{x^{-n}+1}\;(-nx+n+2x-4)]+C \tag{2.11}$$

$$\int P_1^{(x,b)}(x)\sqrt{x^{-n}+1}\;dx=\frac{1}{12(n-6)(n-4)(n-2)}[x\{2n(n^2-6n+8)x^2$$

$$\times{}_2F_1\left(\frac{1}{2},-\frac{3}{n};\frac{n-3}{n};-x^{-n}\right)-3(2(bn(n^2-10n+24)\,{}_2F_1\left(\frac{1}{2},-\frac{1}{n};\frac{n-1}{n};-x^{-n}\right)$$

$$+2\sqrt{x^{-n}+1}(b(n-6)(n(x-1)-2x+4)+(n-2)x(n(x+3)-2(2x+9))))$$

$$-(b+3)n(n^2-8n+12)x\,{}_2F_1\Big(\frac{1}{2},-\frac{2}{n};\frac{n-2}{n};-x^{-n}\Big)))+C \qquad (2.12)$$

$$\int E_1(x)\sqrt{1+x^{-n}}\ dx = \frac{1}{2(n-4)(n-2)}x[(n-2)nx\,{}_2F_1\Big(\frac{1}{2},-\frac{2}{n};\frac{n-2}{n};-x^{-n}\Big)$$

$$-(n-4)n\,{}_2F_1\Big(\frac{1}{2},-\frac{1}{n};\frac{n-1}{n};-x^{-n}\Big)\ +2\sqrt{x^{-n}+1}\ \ (-2nx+n+4x-4)]+C \qquad (2.13)$$

where C is the integral constant.

REFERENCES

[1] Abramowitz, Milton; Stegun, Irene A.(**1965**), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York: Dover.
[2] Andrews, L.C.(**1992**) ; *Special Function of mathematics for Engineers,Second Edition*, McGraw-Hill Co Inc., New York.
[3] Arfken, G. ; *Mathematical Methods for Physicists*, Orlando, FL: Academic Press.
[4] Bells, Richard, Wong, Roderick; *Special Functions, A Graduate Tex*t, Cambridge Studies in Advanced Mathematics, **2010**.
[5] Prudnikov, A. P., Marichev, O. I., and Brychkov, Yu. A. ; *Integrals and Series, Vol. 3: More Special Functions.*, NJ:Gordon and Breach, **1990**.
[6] Rivlin, T. J. ;*Chebyshev Polynomials.*, New York: Wiley, **1990**.
[7] Wikipedia(**2013**,October), Bernoulli Polynomials.
Retrived from website: http://en.wikipedia.org/wiki/Bernoulli_polynomials.
[8] Wikipedia(**2013**,October), Gegenbaur Polynomials.
Retrived from website: http://en.wikipedia.org/wiki/Gegenbauer_polynomials.
[9] WolframMathworld(**2013**,October), Harmonic Number .
Retrived from website: http://mathworld.wolfram.com/HarmonicNumber.html.
[10] WolframMathworld(**2013**,October), LaguerrePolynomials.
Retrived from website: http://mathworld.wolfram.com/LaguerrePolynomial.html.
[11] WolframMathworld(**2013**,October), Hermite Polynomials.
Retrived from website: http://mathworld.wolfram.com/HermitePolynomial.html.
[12] WolframMathworld(**2013**,October), Legendre Polynomial.
Retrived from website: http://mathworld.wolfram.com/LegendrePolynomial.html.
[13] WolframMathworld(**2013**,October), Chebyshev Polynomial of the first kind.
Retrived from website: http://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html.
[14] WolframMathworld(**2013**,October), Lucas polynomials.
Retrived from website: http://mathworld.wolfram.com/LucasPolynomials.html.