Some curious results involving certain polynomials

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ABSTRACT

In this paper we have developed some curious results involving certain polynomials.

Key Words: Hypergeometric function, Laplace Transform, Lucas Polynomials, Gegenbauer polynomials, Harmonic number, Bernoulli Polynomial, Hermite Polynomials.

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INTRODUCTION

We have the generalized Gaussian hypergeometric function of one variable

\[ \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \ldots (a_A)_k}{(b_1)_k(b_2)_k \ldots (b_B)_k} \frac{x^k}{k!} \]  \tag{1.1}

where the parameters \( b_1, b_2, \ldots, b_B \) are neither zero nor negative integers and \( A, B \) are non-negative integers. The series converges for all finite \( z \) if \( A \leq B \), converges for \(|z|<1\) if \( A=B+1 \), diverges for all \( z \), \( z \neq 0 \) if \( A > B+1 \).

Laplace Transform

The Laplace Transform of a function \( f(t) \), defined for all real numbers \( t \geq 0 \), is the function \( F(s) \), defined by:

\[ F(s) = L\{f(t)\}(s) = \int_0^{\infty} e^{-st} f(t) dt \] \tag{1.2}

The parameter \( s \) is a complex number:

\[ s = \sigma + i\omega, \] with real number \( \sigma \) and \( \omega \).

Jacobi Polynomials

In mathematics, Jacobi polynomials (occasionally called hypergeometric polynomials) are a class of classical orthogonal polynomials. They are orthogonal with respect to the weight

\[(1-x)^\alpha (1+x)^\beta\]

on the interval [-1, 1]. The Gegenbauer polynomials, and thus also the Legendre, Zernike and Chebyshev polynomials, are special cases of the Jacobi polynomials.

The Jacobi polynomials were introduced by Carl Gustav Jacob Jacobi.

The Jacobi polynomials are defined via the hypergeometric function as follows:
\[ p_n^{(\alpha, \beta)}(z) = \frac{(\alpha+1)_n}{n!} F_1(-n,1+\alpha+\beta+n;\alpha+1;\frac{1-z}{2}), \]  

(1.3)

Where \((\alpha+1)_n\) is Pochhammer's symbol. In this case, the series for the hypergeometric function is finite, therefore one obtains the following equivalent expression:

\[ p_n^{(\alpha, \beta)}(z) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)} (\frac{z-1}{2})^m \]  

(1.4)

**Lucas Polynomials**

The Lucas polynomials are the w-polynomials obtained by setting \(p(x) = x\) and \(q(x) = 1\) in the Lucas polynomials sequence. It is given explicitly by

\[ L_n(x) = 2^n[(\sqrt{x^2+4})^n + (x + \sqrt{x^2+4})^n] \]  

(1.5)

The first few are

\[
\begin{align*}
L_0(x) &= x \\
L_2(x) &= x^2 + 2 \\
L_3(x) &= x^3 + 3x \\
L_4(x) &= x^4 + 4x^2 + 2
\end{align*}
\]  

(1.6)

**Generalized Harmonic Number**

The generalized harmonic number of order \(n\) of \(m\) is given by

\[ H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m} \]  

(1.7)

In the limit of \(n \to \infty\) the generalized harmonic number converges to the Riemann zeta function

\[ \lim_{n \to \infty} H_n^{(m)} = \zeta(m) \]  

(1.8)
Bernoulli Polynomial

In mathematics, the Bernoulli polynomials occur in the study of many special functions and in particular the Riemann zeta function and Hurwit zeta function. This is in large part because they are an Appell sequence, i.e. a Sheffer sequence for the ordinary derivative operator. Unlike orthogonal polynomials, the Bernoulli polynomials are remarkable in that the number of crossing of the x-axis in the unit interval does not go up as the degree of the polynomials goes up. In the limit of large degree, the Bernoulli polynomials, appropriately scaled, approach the sine and cosine functions.

Explicit formula of Bernoulli polynomials is

\[ B_n(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} b_{n-k} x^k \], for \( n \geq 0 \), where \( b_k \) are the Bernoulli numbers.

The generating function for the Bernoulli polynomials is

\[ \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \] (1.9)

Gegenbauer polynomials

In Mathematics, Gegenbauer polynomials or ultraspherical polynomials \( C_n^{(\alpha)}(x) \) are orthogonal polynomials on the interval \([-1,1]\) with respect to the weight function \( (1-x^2)^{\alpha-1/2} \). They generalize Legendre polynomials and Chebyshev polynomials, and are special cases of Jacobi polynomials. They are named after Leopold Gegenbauer. Explicitly,

\[ C_n^{(\alpha)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)k!(n-2k)!} (2x)^{n-2k} \] (1.10)
Laguerre polynomials

The Laguerre polynomials are solutions $L_n(x)$ to the Laguerre differential equation

$$xy'' + (1 - x)y' + \lambda y = 0,$$

which is a special case of the more general associated Laguerre differential equation, defined by

$$xy'' + (\nu + 1 - x)y' + \lambda y = 0,$$

where $\lambda$ and $\nu$ are real numbers with $\nu=0$.

The Laguerre polynomials are given by the sum

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{n!}{(n-k)!} x^k$$

(1.11)

Hermite polynomials

The Hermite polynomials $H_n(x)$ are set of orthogonal polynomials over the domain $(-\infty, \infty)$ with weighting function $e^{-x^2}$.

The Hermite polynomials $H_n(x)$ can be defined by the contour integral

$$H_n(z) = \frac{n!}{2\pi i} \oint e^{-z^2 + 2\pi i t} t^{-n-1} dt,$$

Where the contour incloses the origin and is traversed in a counterclockwise direction.
The Legendre polynomials, sometimes called Legendre functions of the first kind, Legendre coefficients, or zonal harmonics (Whittaker and Watson 1990, p. 302), are solutions to the Legendre differential equation. If \( l \) is an integer, they are polynomials. The Legendre polynomials \( P_n(x) \) are illustrated above for \( x \in [-1, 1] \) and \( n=1, 2, ..., 5 \).

The Legendre polynomials \( P_n(x) \) can be defined by the contour integral

\[
P_n(z) = \frac{1}{2\pi i} \oint (1 - 2tz + t^2)^{\frac{1}{2}} t^{n-1} dt,
\]

where the contour encloses the origin and is traversed in a counterclockwise direction (Arfken 1985, p. 416).
The second solution $Q_1(x)$ to the Legendre differential equation. The Legendre functions of the second kind satisfy the same recurrence relation as the Legendre polynomials.

The first few are

$$
Q_0(x) = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right),
Q_1(x) = \frac{x}{2} \ln \left( \frac{1+x}{1-x} \right) - 1,
Q_2(x) = \frac{3x^2 - 1}{4} \ln \left( \frac{1+x}{1-x} \right) - \frac{3x}{2}
$$

Chebyshev polynomial of the first kind

The Chebyshev polynomials of the first kind are a set of orthogonal polynomials defined as the solutions to the Chebyshev differential equation and denoted $T_n(x)$. They are used as an approximation to a least squares fit, and are a special case of the Gegenbauer polynomial with $\phi = 0$. They are also intimately connected with trigonometric multiple-angle formulas.

The Chebyshev polynomial of the first kind $T_n(z)$ can be defined by the contour integral

$$
T_n(z) = \frac{1}{4\pi i} \int_{|t|=1} \frac{(1-t^2)^{n-1}}{1-2tz+t^2} dt,
$$

where the contour encloses the origin and is traversed in a counterclockwise direction (Arfken 1985, p. 416). The first few Chebyshev polynomials of the first kind are

$$
T_0(x) = 1,
T_1(x) = x,
T_2(x) = 2x^2 - 1,
T_3(x) = 4x^3 - 3x.
$$

A beautiful plot can be obtained by plotting $T_n(x)$ radially, increasing the radius for each value of $n$, and filling in the areas between the curves (Trott 1999, pp. 10 and 84).
The Chebyshev polynomials of the first kind are defined through the identity

\[ T_n(\cos \theta) = \cos n\theta. \]

Chebyshev polynomial of the second kind

A modified set of Chebyshev polynomials defined by a slightly different generating function. They arise in the development of four-dimensional spherical harmonics in angular momentum theory. They are a special case of the Gegenbauer polynomial with \(\alpha = 1\). They are also intimately connected with trigonometric multiple-angle formulas.

The first few Chebyshev polynomials of the second kind are

\[
\begin{align*}
U_0(x) &= 1 \\
U_1(x) &= 2x \\
U_2(x) &= 4x^2 - 1 \\
U_3(x) &= 8x^3 - 4x
\end{align*}
\]

(1.16)
The Euler polynomial $E_n(x)$ is given by the Appell sequence with
\[ g(t) = \frac{1}{2} (e^t + 1) , \]
giving the generating function
\[ \frac{2e^n}{e^t + 1} \equiv \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} . \quad (1.17) \]

The first few Euler polynomials are
\[
\begin{align*}
E_0(x) &= 1 \\
E_1(x) &= x - \frac{1}{2} \\
E_2(x) &= x^2 - x \\
E_3(x) &= x^3 - \frac{3}{2} x^2 + \frac{1}{4}
\end{align*}
\quad (1.18)
\]

1. **MAIN RESULTS**

\[
\begin{align*}
\int L_1(x) \sqrt{1 + x^{-n}} \, dx &= \frac{x^2 \left( n_2 F_1 \left( \frac{1}{2}, -\frac{2}{n}; -\frac{n-2}{n}; -x^{-n} \right) - 4\sqrt{x^{-n} + 1} \right)}{2(n-4)} + C \\
\int H_1^{(x)} \sqrt{1 + x^{-n}} \, dx &= \frac{n x_2 F_1 \left( \frac{1}{2}, -\frac{1}{n}; -\frac{n-1}{n}; -x^{-n} \right) - 2x\sqrt{x^{-n} + 1}}{(n-2)} + C \\
\int B_1(x) \sqrt{1 + x^{-n}} \, dx &= \frac{1}{2(n-4)(n-2)} x l(n-2) n x_2 F_1 \left( \frac{1}{2}, -\frac{2}{n}; -\frac{n-2}{n}; -x^{-n} \right)
\end{align*}
\]
\[-(n-4)n_2F_1\left(\frac{1}{2}, 1, -\frac{1}{n}; \frac{n-1}{n}, -x^n\right) + 2\sqrt{x^n+1} \left(-2nx + 4x - 4\right)\] + C

\[\int C(x)\sqrt{1+x^n} \, dx = \frac{x^2(n_2F_1\left(\frac{1}{2}, 1, -\frac{2}{n}; \frac{n-1}{n}, -x^n\right) - 4\sqrt{x^n+1}}{(n-4)} + C \quad (2.3)\]

\[\int F_1(x)\sqrt{1+x^n} \, dx = \frac{x^2(n_2F_1\left(\frac{1}{2}, 1, -\frac{2}{n}; \frac{n-1}{n}, -x^n\right) - 4\sqrt{x^n+1}}{2(n-4)} + C \quad (2.4)\]

\[\int L_{(n)}(s)\sqrt{1+x^n} \, dx = \frac{nx_2 F_1\left(\frac{1}{2}, -\frac{1}{n}; \frac{n-1}{n}, -x^n\right) - 2x\sqrt{x^n+1}}{(n-2)s} + C \quad (2.5)\]

\[\int P_0(x)\sqrt{1+x^n} \, dx = \frac{x^2(n_2F_1\left(\frac{1}{2}, 1, -\frac{2}{n}; \frac{n-1}{n}, -x^n\right) - 4\sqrt{x^n+1}}{2(n-4)} + C \quad (2.6)\]

\[\int T_0(x)\sqrt{1+x^n} \, dx = \frac{x^2(n_2F_1\left(\frac{1}{2}, 1, -\frac{2}{n}; \frac{n-1}{n}, -x^n\right) - 4\sqrt{x^n+1}}{2(n-4)} + C \quad (2.7)\]

\[\int U_0(x)\sqrt{1+x^n} \, dx = \frac{x^2(n_2F_1\left(\frac{1}{2}, 1, -\frac{2}{n}; \frac{n-1}{n}, -x^n\right) - 4\sqrt{x^n+1}}{(n-4)} + C \quad (2.8)\]

\[\int H_0(x)\sqrt{1+x^n} \, dx = \frac{x^2(n_2F_1\left(\frac{1}{2}, 1, -\frac{2}{n}; \frac{n-1}{n}, -x^n\right) - 4\sqrt{x^n+1}}{(n-4)} + C \quad (2.9)\]

\[\int L_0(x)\sqrt{1+x^n} \, dx = \frac{1}{2(n-4)(n-2)}x(l(n-2)nx_2 F_1\left(\frac{1}{2}, -\frac{2}{n}; \frac{n-1}{n}, -x^n\right) - 2(n-4)n_2F_1\left(\frac{1}{2}, -\frac{1}{n}; \frac{n-1}{n}, -x^n\right) + 4\sqrt{x^n+1} \left(-nx + n + 2x - 4\right)\] + C

\[\int P_0^{(x,b)}(x)\sqrt{x^n+1} \, dx = \frac{1}{12(n-6)(n-4)(n-2)} [x(2n(n^2 - 6n + 8)x^2 \times x_2 F_1\left(\frac{1}{2}, 1, -\frac{3}{n}; \frac{n-3}{n}, -x^n\right) - 3(2bn(n^2 - 10n + 24))_2 F_1\left(\frac{1}{2}, -\frac{1}{n}; \frac{n-1}{n}, -x^n\right) + 2\sqrt{x^n+1}(b(n-6)(n(x-1) - 2x + 4) + (n-2)x(n(x+3) - 2(2x+9)))}]

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\[(b+3)n(n^2-8n+12)x_2 \ _1F_1 \left( \frac{1}{2}, -\frac{2}{n}; \frac{n-2}{n}; -x^{-n} \right) + C \]

\[\int E_4(x)\sqrt{1+x^n} \ dx = \frac{1}{2(n-4)(n-2)}x_1(n-2)nx_2 \ _1F_1 \left( \frac{1}{2}, -\frac{2}{n}; \frac{n-2}{n}; -x^{-n} \right) \]

\[= -(n-4)n_2 \ _1F_1 \left( \frac{1}{2}, -\frac{1}{n}; \frac{n-1}{n}; -x^{-n} \right) + 2\sqrt{x^{-n} + 1} \ (-2nx + n + 4x - 4) + C \]

where \( C \) is the integral constant.

REFERENCES