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Study of the mellin integral transform with applications in statistics And probability

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ABSTRACT

The Fourier integral transform is well known for finding the probability densities for sums and differences of random variables. We use the Mellin integral transforms to derive different properties in statistics and probability densities of single continuous random variable. We also discuss the relationship between the Laplace and Mellin integral transforms and use of these integral transforms in deriving densities for algebraic combination of random variables. Results are illustrated with examples.

Keywords: Laplace, Mellin and Fourier Transforms, Probability densities, Random Variables and Applications in Statistics.
AMS Mathematical Classification: *44F35, 44A15, 44A35, 44A12, 43A70.*

INTRODUCTION

The aim of this paper , we define a random variable (RV) as a value in some domain , say $\Re$, representing the outcome of a process based on a probability laws .By the information above the probability distribution ,we integrate the probability density function (p d f)in the case of the Gaussian , the p d f is

$$f(x)= \frac{1}{\sigma\sqrt{2\pi}} E^{-\frac{1}{2}(\frac{X-\pi}{\sigma})^2}, -\infty < x < \infty$$

when μ is mean and σ^2 is the variance.

We define the p d f for X+Y and XY ,where X and Y are the R Vs, by using the brief background on probability theory and see the convolution by using Laplace – Mellin integral transforms.

In this paper we define Mellin integral transform , continuous random variable for X and its p d fs , continuous distribution function, expectations and moments about origin and mean for independent CRVs X , mode , median , quartiles , deciles , percentiles , skew ness (by using mean and mode and also by using quartiles) , kurtosis (by using moments) **:** Transform for the sum of the random variables, Convolution Algebra on $L^1(\Re)$,: The Mellin integral transform, and relation with Laplace integral transform relation in between expected values and moments of CRVs X ,one dimensional continuous random variable and its p d fs , marginal density functions, theorems of addition and multiplication of CRVs X and Y, relations in between expected values of CRVs X and Y and Mellin integral transform.

3.1.2: Terminology

To avoid confusion , it is necessary to mention a few cases in which the terminology used in probability theory * "Distribution",(or "law") in probability theory means a function that assigns a probability $0 \le p \le 1$ to every Borel subset of $\Re$; not a "generalized function" ,as in the Schwartz theory of distribution.

* For historical reasons going back to Henri Poincard , the term " characteristic function " in probability theory refers to as integral transform of a p d f, not to what mathematicians usually refer to as the characteristic function .For that concept probability theory uses " indicator function " ,symbolized I; e.g. $I_{(0,1)}(x)$ is 1 for x$\in [0,1]$ and 0 elsewhere .In this paper as will not use the term "characteristic function " at all.

* We will be talking about p d fs being in $L^1(\Re)$, and this should be taken in the ordinary mathematics a sense of a function on $\Re$ which is absolutely integrable . More commonly , probabilists talk about random variables being in L^1, L^2 , etc.

Which is quite different -in terms of a p d f f, it means that $\int |x| f(x)dx, \int |x|^2 f(x)dx$, etc exist and are finite. It would require an excursion into measure theory to explain why this makes sense , suffice it to say in the latter case we should really say something like "$L^1(\Omega, F, P)$" , which is not at all the same as $L^1(\Re)$.

3,1.3 : PROBABILITY BACKGROUND

Probability theory starts with the idea of the outcome of some process , which is mapped to a domain (i.e. $\Re$)by a random variable say X ,we just think of $x \in \Re$

As a 'realization' of X , with a probability law or distribution which tales us how much probability is associated with any interval [a ,b]$\subset \Re$ .How much is given by a number $0 \le p \le 1$

Formally , probabilities are implically defined by their role in the axioms of probability theories .A probability law in $\Re$ can be represented by its density or p d f which is a continuous function f(x) with the probability that the probabilities of finding x in [a , b] i.e. $P(x \in [a,b]) = \int_a^b f(x)dx$,it gives the probability "mass" per unit length ,which is integrated to measure the total mass in an interval .

We define characteristics of a probability measure on $\Re$

1: for any [a , b] , $0 \le P(x \in [a,b]) \le 1$

2: $P[x \in (-\infty, \infty)] = 1$

3: if $[a,b] \cap [c,d] = \phi$, then $P(x \in [a,b] \cap [c,d]) = P(x \in [a,b]) + P(x \in [c.d])$

From these properties and general properties of integral it follows that is f is a continuous p d f , then $f(x) \ge 0$

and $\int_{-\infty}^{\infty} f(x)dx = 1$

For any RV X ,with p d f f, it mean $\overline{X} = \mu = \int_{-\infty}^{\infty} xf(x)dx$,,this is usually designated by E(X),the expectation or expected value of X ,the variance of X is

$$E\{(x-\mu)^2\} = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

A frequently used modes for such RVs is the Mellin distribution , with p d f.

f(x)= $\frac{1}{\Gamma(r)} x^{r-1}e^{-x}$,if x>0,0 otherwise ,when $\Gamma(r) = \int_{0}^{\infty} x^{r-1}e^{-x}dx$,where x^{r-1} is the Mellin kernel foe r>0 is a parameter .We see expectation ,mean , variance , moments ,mode, median , skew ness , kurtosis ,etc.

Independence is profoundly important in probability theory ,and is mainly what saves probability from being 'merely' an application of measure theory .For the purpose of this paper , an intuitive definition suffices :two random variables X , Y are independent if the occurrence or nonoccurrence of an event $X \in [a,b]$ does not affect the probability of an event $Y \in [a,b]$ and vice versa .Computationally ,the implication is that "independence means multiply" i. e. if X ,Y are independent,

$$P(x \in [a,b] \,\&\, y \in [a,b]) = P(X \in [a,b]P(Y \in [a,b]$$

In this paper , we will only consider independent RVs .If X ,Y are RVs ,by substituting $[-\infty,\infty]$ for either of the intervals of integration , it is seen that

$$\int_a^b f_X(x)dx \int_{-\infty}^{\infty} f_Y(y)dy = \int_a^b \int_{-\infty}^{\infty} f_X(x) f_Y(y)dxdy = \int_a^b \int_{-\infty}^{\infty} f_{XY}(x,y)dxdy$$

$$= \int_a^b \int_{-\infty}^{\infty} f(x,y)dxdy = \int_a^b [\int_{-\infty}^{\infty} f(x,y)dy]dx = \int_a^b [\int_{-\infty}^{\infty} f_X(x)dx$$

where $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$,this is true and $f_Y(y)$ is the marginal density of Y.

3.1.4: Preliminary Results

3.1.4.1: Dfefinitions

The Mellin integral transform of the function f(x) with its kernel x^{r-1} and r>0 is the parameter , is denoted by M[f(x),r] and defined as

$$M\,[f(x), r] = \int_0^{\infty} x^{r-1} f(x)dx, 0<x<\infty, r>0,$$

whenever this integral is exist. This is an important integral transform ,whose use in statistics is related to recovering of probability distributions, product or quotient of independent nonnegative continuous random variables.

One of the main problems arising in the applications is that of inverting Mellin integral transform ,i. e. the determination of the original function f(x) from the transform

M [f(x), r] .This problem , formally solved by the inverse formula

f(x)= $\frac{1}{2\pi i}\int_{c-i\infty}^{c\_i\infty} x^{-r} M[f(x)mr]dr$,can not be solved ,in most of cases, analytically. Therefore the only possible of inverting the transformed functions is by numerical means.

3.1.4.2: Continuous Random Variable

If X is an random variable which can take all values (I .e. infinite number of values)in an interval , then X is called a **continuous random variable.**

Its probability distribution is known as **continuous probability distribution** .

If function f(x) is said to be a function of continuous random variable X ,if it satisfies the following conditions

(1) f(x)≥ 0 and (2) $\int_{-\infty}^{\infty} f(x)dx = 1$

3.1.4.3: Probability Density Function of Continuous Random Variable

A continuous function y=f(x) such that

(1) f (x) is integrable

(2) $\int_{a}^{b} f(x)dx = 1$ if X lies in [a ,b] and

(3) $\int_{\alpha}^{\beta} f(x)dx = P(\alpha \leq x \leq \beta)$, where $a < \alpha < \beta < b$ is called **probability function of a continuous random variable X.**

Thus for a continuous random variable

$$\int_{\alpha}^{\beta} f(x)dx = P(\alpha \leq x \leq \beta)$$

Clearly $\int_{\alpha}^{\beta} f(x)dx$ represents the area under the curve f(x) ,the x-axis and the ordinates x=α and x=β

3.1.4.4: Continuous Distribution Function

Probability distribution of X or the probability density function of X helps os to find the probability that X will be within a given interval [a ,b] i.e.

$$P(a \leq x \leq b) - \int_{a}^{b} f(x)dx,$$

other conditions being satisfied.

If X is a continuous random variable , having the probability density function f(x) then the function

$$P(x) = P(X \leq x) = \int_{-\infty}^{\infty} f(t)dt$$

$$P(x) = P(X \le x) = \int_{-\infty}^{\infty} f(x)dx \qquad , -\infty \le x \le \infty$$

is called distribution function or **cumulative distribution function of the continuous random variable X.**

3.1.4.5: Probabilities of Distribution Function F(x) of a Continuous Random Variable

1: The function F(x) is defined for every real number x

2: since F(x) denotes probability and probability of x lies between o and 1

$$0 \le F(x) \le 1$$

3: F(x) is a non-decreasing function which means if

$$x_{11} \le x_2 \,,\, F(x_1) \le F(x_2)$$

3.1.4.6: The Derivative of F(x) :

F(x) exists at all points (except perhaps at a finite number of points) and is equal to the probability density function f(x),

$$F(x) = \frac{d}{dx} f(x) = f(x) \ge 0$$

Provided derivative exists.

If F(x) is a distribution function of a continuous random variable then

$$P(a \le x \le b) = F(b) - F(a)$$

3.1.5: Main Results

3.1.5.1: Expectation and Moments about origin by using MIT DMIT

The expectation of continuous random variable X is denoted by E[X] and defined as

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

For the Mellin Integral Transform the Probability Density Function is

$$f(x)=\frac{1}{\Gamma(r)} x^{r-1}e^{-x} \text{ , r>0}$$

where x^{r-1} is the Mellin kernel. then $\int_{-\infty}^{\infty} f(x)dx = 1$

For the Mellin Transform

$$\int_0^{\infty} f(x)dx = \int_0^{\infty} \frac{1}{\Gamma(r)} x^{r-1}e^{-x}dx = \frac{1}{\Gamma(r)} \int_0^{\infty} x^{r-1}e^{-x}dx$$

$$= \frac{\Gamma(r)}{\Gamma(r)} = 1 \text{ , where } \int_{-\infty}^{\infty} f(x)dx = 1$$

If f(x) = $\frac{1}{\Gamma(r)} e^{-x}$ is the function of continuous random variable X, then

$$M\,[f(x),s] = \int_0^{\infty} x^{r-1} f(x) = \int_0^{\infty} \frac{1}{\Gamma(r)} x^{r-1}e^{-x}dx = \frac{1}{\Gamma(r)} \int_0^{\infty} x^{r-1}e^{-x}dx$$

$$= \frac{\Gamma(r)}{\Gamma(r)} = 1 = \mu'_{x0}$$

M [f(x), s] =1, is the moment about origin , denoted by $\mu_0^{'}$, $\mu_0^{'}$=1

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} x^{r-1}\frac{1}{\Gamma(r)}e^{-x}dx = \frac{1}{\Gamma(r)}\int_{-\infty}^{\infty} x^{r}e^{-x}dx$$

$$= \frac{1}{\Gamma(r)}\int_{0}^{\infty} x^{r+1-1}e^{-x} = \frac{1}{\Gamma(r)}\Gamma(r+1) = \frac{(r+1)\Gamma(r)}{\Gamma(r)} = (r+1) = \mu_1^{'}$$

$E[X] = \mu_1^{'} = (r+1)$

$$E[X^2] = \int_{0}^{\infty} x^2 f(x)dx = \int_{0}^{\infty} x^2\frac{1}{\Gamma(r)}x^{r-1}e^{-x}dx = \frac{1}{\Gamma(r)}\int_{0}^{\infty} x^2x^{r-1}e^{-x}dx$$

$$= \frac{1}{\Gamma(r)}\int_{0}^{\infty} x^{r+1}e^{-x}dx = \frac{1}{\Gamma(r)}\int_{0}^{\infty} x^{r+2-1}e^{-x}dx$$

$$= \frac{\Gamma(r+2)}{\Gamma(r)} = \frac{(r+1)(r+2)\Gamma(r)}{\Gamma(r)} = (r+1)(r+2) = \mu_2^{'}$$

$E[X^2] = \mu_2^{'} = (r+1)(r+2)$

$$E[X^3] = \int_{0}^{\infty} x^3 f(x)dx = \int_{0}^{\infty} x^3\frac{1}{\Gamma(r)}x^{r-1}e^{-x}dx = \frac{1}{\Gamma(r)}\int_{0}^{\infty} x^3x^{r-1}e^{-x}dx$$

$$= \frac{1}{\Gamma(r)}\int_{0}^{\infty} x^3x^{r+3-1}e^{-x}dx$$

$$= \frac{\Gamma(r+3)}{\Gamma(r)} = \frac{(r+1)(r+2)(r+3)\Gamma(r)}{\Gamma(r)} = (r+1)(r+2)(r+3) = \mu_3^{'}$$

$E[X^3] = \mu_3^{'} = (r+1)(r+2)(r+3)$

$$E[X^4] = \int_{0}^{\infty} x^4 f(x)dx = \int_{0}^{\infty} x^4\frac{1}{\Gamma(r)}x^{r-1}e^{-x}dx \; \frac{1}{\Gamma(r)}\int_{0}^{\infty} x^4x^{r-1}e^{-x}dx$$

$$= \frac{1}{\Gamma(r)}\int_{0}^{\infty} x^{r+4-1}e^{-x}dx$$

$$= \frac{\Gamma(r+4)}{\Gamma(r)} = \frac{(r+1)(r+2)(r+3)(r+4)\Gamma(r)}{\Gamma(r)}$$

$= (r+1)(r+2)(r+3)(r+4) = \mu_4^{'}$

$E[X^4] = \mu_4^{'} = (r+1)(r+2)(r+3)(r+4)$

3.1.5.2: The Mellin Integral Transform and Moments (Moments about origin)

If X is a cintineous random variable, then the expectations are as follows

1: $E[x^{r-1}] = \int_{0}^{\infty} x^{r-1} f(x)dx$ =M[f(x),r] $= \mu_{x0}^{'}$=1

2: $E[x^r]=\int_0^\infty x^r f(x)dx=\int_0^\infty x^{r+1-1} f(x)dx$ =M[f(x),r+1]= μ'_{x1} =(r+1)

3: $E[x^{r+1}]=\int_0^\infty x^{r+1} f(x)dx=\int_0^\infty x^{r+2-1} f(x)dx$ =M[f(x),r+2] = μ'_{x2} =(r+1)(r+2)

4: $E[x^{r+2}]=\int_0^\infty x^{r+2} f(x)dx=\int_0^\infty x^{r+3-1} f(x)dx$ =M[f(x),r+3]= μ'_{x3} =(s+1)(s+2)(s+3)

5: $E[x^{r+3}]=\int_0^\infty x^{r+3} f(x)dx=\int_0^\infty x^{r+4-1} f(x)dx$ =M[f(x),r+4]

= μ'_{x4} =(s+1)(s+2)(s+3)(s+4)

If f(x) = $\frac{1}{\Gamma(s)}$ is the function of continuous random variable Y , then

6: $E[Y^{r-1}]=\int_0^\infty y^{s-1} f(y)dy$ =M[f(y),r]= μ'_{y0} =1

7: $E[Y^r]=\int_0^\infty y^s f(y)dy=\int_0^\infty x^{s+1-1} f(x)dx$ =M[f(y),s+1]= μ'_{y1} =(s+1)

8: $E[Y^{r+1}]=\int_0^\infty y^{s+1} f(y)dy=\int_0^\infty y^{s+2-1} f(y)dy$ =M[f(y),s+2] = μ'_{y2} =(s+1)(s+2)

9: $E[Y^{r+2}]=\int_0^\infty y^{s+2} f(y)dy=\int_0^\infty y^{r+3-1} f(y)dy$ =M[f(y),s+3]= μ'_{y3} =(s+1)(s+2)(s+3)

10: $E[Y^{r+3}]=\int_0^\infty y^{s+3} f(y)dy=\int_0^\infty y^{s+4-1} f(y)dy$ =M[f(y),s+4]

= μ'_{y4} =(s+1)(s+2)(s+3)(s+4)

3.1.5.3: Moments Abour Mean , Variance , Skewness and Kurtosis

The varience of the random variable X is denoted by V(X) and defined as

1: Varience of X=V(X)

=E[(X-m)^2] = E[X^2]-(E[X])^2 = μ_2

= $\mu_{x2}=\mu'_{x2}-(\mu''_{x1})^2$ =(r+1)(r+2)-(r+1)^2 =(r+1)(r+2-r-1)

V(X) =(r+1)= μ_{x2}

The other moments about mean are obtained using relations in between moments about a origin and moments about mean.

2: $\mu_{x3}=\mu'_{x3}-3\mu'_{x2}\mu'_{x1}+2(\mu'_{x1})^3$

=(r+1)(r+2)(r+3)-3(r+1)(r+2)(r+1)+2(r+1)^3

=(r+1)[(r+2)(r+3)-3(r+1)(r+2)+2(r+1)^2]

=(r+1)($r^2+5r+6-3r^2-9r-6+2r^2+4r+2)$

μ_{x3} =2(r+1)

3: $\mu_{x4} = \mu'_{x4} - 4\mu'_{x3}\mu'_{x1} + 6\mu'_{x2}(\mu'_{x1})^2 - 3(\mu'_{x1})^4$

=(r+1)(r+2)(r+3)(r+4)-4(r+1)(r+2)(r+3)(r+1)+6(r+1)(r+2)(r+1) ^{2} -3(r+1) ^{4}

=(r+1)[$(r+2)(r^2+7r+12-4(r+2)(r^2+4r+3)+6(r+2)(r^2+2r+1)-3(r^3+3r^2+3r+1)$]

=(r+1)($r^3+7r^2+12r+2r^2+14r+24$ - $4r^3-16r^2-12r-8r^2-32r-24$

+ $6r^3+12r^2+6r+12r^2+24r+12-3r^3-9r^2-9r-3)$

=9(r+1)

μ_{x4} =9(r+1)

3.1.5.4: Measure of Skewness ($\beta_1\ \gamma_1$)

Karl Pearson's defined the four coefficients based on moments about mean
These are used to measure the skewness and kurtosis
By using the moments about mean , we define the Karl Pearson's skewness and as follows

skewness $= \gamma_1 = \sqrt{\beta_1} = \sqrt{\frac{\mu_3^2}{\mu_2^3}} = \sqrt{\frac{8(r+1)^2}{(r+1)^3}} = \sqrt{\frac{8}{r+1}}$

If β_1 =0 , the distribution is symmetric

If β_1 <0 , the distribution is negative skew

If β_1 >0 , the distribution is positive skew

For r= 1 to n , β_1 >0 ,the distribution is positive skew for Mellin integral transform.

3,1.5.5: Measure of Kurtosis (β_2, γ_2)

Kurtosis= $\gamma_2 = \beta_2 - 3 = \frac{\mu_3}{\mu_2^2}$ -3= $\frac{9(r+1)}{(r+1)^2}$ -3= $\frac{9}{r+1} - 3$

(1) If γ_2 =0, the distribution is normal or mesokurtosis

for β_2 =3 when r=2

If γ_2 >0, the distribution is more peaked or leptokurtosis

for β_2 >3 when r=1

If γ_2 <0, the distribution is more flat or platykurtosis

for β_2 <3 when $r \geq 3$

3.1.5.6: Mode

If f(x) be a function of continuous random variable X , then $\frac{dy}{dx} = f'(x) = 0$

We get values of X i.e. $X_1, X_2, \ldots,$ X_n , and if $[\frac{d^2y}{dx^2}]_{X=X_{1_1}}$ <0 , then $X = \overline{X}$ is the mode

If $f(x) = \frac{1}{\Gamma(r)} e^{-x} x^{r-1}$ be the contineius function of random variable X, then

$f'(x) = \frac{1}{\Gamma(r)}[x^{r-1}(-e^{-x}) + (r-1)x^{r-2}e^{-x}]$

$$=\frac{1}{\Gamma(r)}[x^{r-1}(-e^{-x})+(r-1)x^{r-2}e^{-x}]$$

$$=\frac{x^{r-2}e^{-x}}{\Gamma(r)}[-x+(r-1)]$$

$$=\frac{x^{r-2}e^{-x}}{\Gamma(r)}[-x+r-1]$$

f'(x)=0 , then

-x+r-1=0

x=r-1, is the point

$$f''(x)=\frac{-e^{-x}}{\Gamma(r)}[-x^{r-1}+(r-1)x^{r-2}]+\frac{e^{-x}}{\Gamma(r)}[-(r-1)x^{r-2}+(r-1)(r-2)x^{r-3}]$$

$$=\frac{e^{-x}}{\Gamma(r)}[x^{r-1}-(r-1)x^{r-2}-(r-1)x^{r-2}+(r-1)(r-2)x^{r-3}]$$

$$=\frac{e^{-x}}{\Gamma(r)}x^{r-3}[x^2-2(r-1)x+(r-1)(r-2)]$$

$$[f''(x)]_{x=r-1}=\frac{e^{-(r-1)}}{\Gamma(r)}(r-1)^{r-3}[(r-1)^2-2(r-1)(r-1)+(r-1)(r-2)]$$

$$=\frac{e^{-(r-1)}}{\Gamma(r)}(r-1)^{r-3}[(r-1)^2-2(r-1)^2+(r-1)(r-2)]$$

$$=\frac{e^{-(r-1)}}{\Gamma(r)}(r-1)^{r-3}[-(r-1)^2+(r-1)(r-2)]$$

$$=\frac{e^{-(r-1)}}{\Gamma(r)}(r-1)^{r-3}[(r-1)(r-2-r+1)]$$

$$=\frac{e^{-(r-1)}}{\Gamma(r)}(r-1)^{r-3}[-(r-1)]$$

$$=-\frac{e^{-(r-1)}}{\Gamma(r)}(r-1)^{r-2}$$

$$[f''(x)]_{x=r-1}$$

Then f(x) is maximum at x=r-1

The value of the mode is r-1

Mo=x=r-1

3.1.5.7: Median , Quartiles , Deciles, ercentiles, QD., Coeff. Of QD, Bowley's Method , Karl Pearson's Method for Coeffg.pf Skewness

If $f(x)=\frac{1}{\Gamma(r)}e^{-x}x^{r-1}$ be a function of continuous random variable X , and

$\int_0^M f(x)dx=\frac{1}{2}$, then M=Md is said to be Median

$\int_0^M \frac{1}{\Gamma(r)} x^{r-1} e^{-x} dx = \frac{1}{2}$ then $\int_0^M \frac{1}{\Gamma(r)} x^{r-1} e^{-x} dx = \frac{1}{2}$

$\sum_0^M \frac{x^{r-1} e^{-x}}{\Gamma(r)} = \frac{1}{2}$ then $\sum_0^M \frac{M^{r-1} e^{-M}}{\Gamma(r)} = \frac{1}{2}$

$e^{-M} \sum_0^M \frac{M^{r-1}}{\Gamma(r)} = \frac{1}{2}$ then $e^{-M} [\frac{M^{-1}}{\Gamma(0)} + 1 + \frac{M}{1!} + \frac{M^2}{2!} + ---] = \frac{1}{2}$

$e^{-M} [\frac{M^{-1}}{\Gamma(0)}] + e^{-M} [1 + \frac{M}{1!} + \frac{M^2}{2!} + ---] = \frac{1}{2}$

$e^{-M} M^{-1} + e^{-M} e^{M} = \frac{1}{2}$ then $e^{-M} M^{-1} \mathbf{+1} = \frac{1}{2}$

$e^{-M} M^{-1} = \frac{-1}{2}$ then $e^{-M} = \frac{-1}{2}\mathbf{M}$

$1 - M + \frac{M^2}{2} - . --- = \frac{-1}{2}\mathbf{M}$

Comparing first term on both sides

$1 = \frac{-1}{2}\mathbf{M}$ then

Md=-2 **,** this is the value of the Median

Q1=Quartile N0.1= - 4

Q3=Quartile No.3= - $\frac{-4}{3}$

$Q_r = \frac{-4}{r}$ where r=1, 2, 3

$P_r = \frac{-10}{r}$ whetre r=1, 2, --,9

Quatile Deviation=QD= $\frac{Q3 - Q1}{2} = \frac{\frac{-4}{3} + 4}{2} = \frac{4}{3}$

Coefficient of Q.D= $\frac{Q3 - Q1}{Q3 + Q1} = \frac{\frac{-4}{3} + 4}{\frac{-4}{3} - 4} = \frac{\frac{8}{3}}{\frac{-16}{3}} = \frac{-1}{2}$ =-0.5

Bowley's Method

Skeness =2(Md)-Q1-Q3= 2(-2)+4+ $\frac{4}{3} = \frac{4}{3}$

Coeff.of Skewness = $\frac{2(Md) - Q1 - Q3}{Q3 + Q1}$

$$= \frac{2(-2) + 4 + \frac{4}{3}}{\frac{-4}{3} + 4} = \frac{\frac{4}{3}}{\frac{8}{3}} = \frac{1}{2} = 0.5$$

Kalpearson's Method

Skeness=Mean-Mode =r+1-(r-1)=r+1-r+1=2

Coeff. Of Skewness $= \dfrac{mean - \bmod e}{std} = \dfrac{(r+1)+(r-1)}{r+1} = \dfrac{2}{r+1}$

where $r \geq 1.1$

3.1.5.8: Mean Deviation from Mean

If X be a continuous randon variabl then its p .d. function if

f(x)= $\dfrac{1}{\Gamma(r)} e^{-x} x^{r-1}$

$$MD= \int_{-\infty}^{\infty} \left|X - \overline{X}\right| f(x)dx = \int_{0}^{\infty} \left|X - \overline{X}\right| \frac{1}{\Gamma(r)} e^{-x} x^{r-1} dx$$

$$= \frac{1}{\Gamma(r)} \int_{-\infty}^{\infty} \left|x - \overline{x}\right| e^{-x} x^{r-1} = \frac{1}{\Gamma(r)} [\int_{0}^{\infty} xe^{-x} x^{r-1} dx - \int_{0}^{\infty} \overline{x} e^{-x} x^{r-1} dx]$$

$$= \frac{1}{\Gamma(r)} [\int_{0}^{\infty} e^{-x} x^{r+1-1} dx - \overline{x} \int_{0}^{\infty} e^{-x} x^{r-1} dx] = \frac{1}{\Gamma(r)} [\Gamma(r+1) - \overline{x}\Gamma(r)]$$

$$= \frac{1}{\Gamma(r)} [(r+1)\Gamma(r) - \overline{x}\Gamma(r)] = \frac{\Gamma(r)}{\Gamma(r)} [r+1-\overline{x}]$$

$MD = r+1-\overline{x}$ where r is positive

3.1.5.9: Probability

If $F(x) = \dfrac{d}{dx} f(x)$, and $f(x) \geq 0$ then

$P(a \leq x \leq b) = F(b) - F(a)$ or

$$P(a \leq x \leq b) = \int_{a}^{b} f(x)dx = \int_{a}^{b} x^{r-1} f(x)dx$$

For Mellin Integral Transform

$P(0 \leq x \leq \infty) = \int_{0}^{\infty} x^{r-1} f(x)dx$, where

f(x)= $\dfrac{e^{-x}}{\Gamma(r)}$ is the continuous .function then

$$P(0 \leq x \leq \infty) = \int_{0}^{\infty} \frac{1}{\Gamma(r)} x^{r-1} e^{-x} dx = 1$$

3.1.5.10: The Sum of Random Variables by using MIT

Suppose that the RV X has p d f $f_X(x)$ and Y has p d f $f_Y(y)$ and X and Y are independent .Consider the transformation $\psi : \Re^2 \to \Re^2$ given by

$$\psi(x, y) = (x.x + y) = (x, z)$$

If we can determine the joint density function $f_{XZ}(x,z)$,then the marginal density function $f_z(z) = \int_{\Re} f_{XZ}(x,z)dx = \int_{\Re} f_{XZ}[\psi^{-1}(x,z)]dx$

$$= \int_{\Re} f_{XZ}(x, z-x)dx = \int_{\Re} f_X(x) f_Y(z-z)dx \ ,$$

where X and Y are independent

$$= f_X * f_Y(z)$$

The nest-to-last line above is intuitive it says that we find the density for Z=X+Y by integrating the joint density of X ,Y over all points where X+Y=Z .i.e. Y=Z-X.

By using the Fourier transform [7]

$$\int_{\Re} \frac{1}{\Gamma(r)} x^{r-1} e^{-x} e^{-2\pi i\xi x} dx = \frac{1}{\Gamma(r)} \int_{\aleph} x^{r-1} e^{-(1+2\pi i\xi)x} dx$$

substitute $(1+2\pi i\xi)x = q$,x=$\frac{q}{1+2\pi i\xi}$, dx=$\frac{dq}{1+2\pi i\xi}$,then

$$= \frac{1}{\Gamma(r)} \int_{\aleph} \left(\frac{q}{1+2\pi i\xi}\right)^{r-1} e^{-q} \frac{dq}{1+2\pi i\xi}$$

$$= \frac{1}{\Gamma(r)} \frac{1}{(1+2\pi i\xi)^r} \int_{\Re} q^{r-1} e^{-q} dq$$

$$= \frac{1}{\Gamma(r)} \frac{\Gamma(r)}{(1+2\pi i\xi)^r}$$

$$= \frac{1}{(1+2\pi i\xi)^r}$$

By using Laplace Transform

$$\int_{\Re} \frac{1}{\Gamma(r)} x^{r-1} e^{-x} e^{-\xi x} dx = \frac{1}{\Gamma(r)} \int_{\aleph} x^{r-1} e^{-(1+\xi)x} dx$$

substitute $(1+\xi)x = q$, x=$\frac{q}{1+\xi}$, dx=$\frac{dq}{1+\xi}$,then

$$= \frac{1}{\Gamma(r)} \int_{\aleph} \left(\frac{q}{1+\xi}\right)^{r-1} e^{-q} \frac{dq}{1+\xi} = \frac{1}{\Gamma(r)} \frac{1}{(1+\xi)^r} \int_{\Re} q^{r-1} e^{-q} dq$$

$$= \frac{1}{\Gamma(r)} \frac{\Gamma(r)}{(1+\xi)^r} = \frac{1}{(1+\xi)^r}$$

3.1.5.11: The Product of Random Variables by using MIT

1: Convolution Algebra On $L^1(\Re)$

The general notion of an algebra is a collection of entries closed under operations that "look like" addition and multiplication of numbers .In the context of function spaces (in particular $L^1(\Re)$,which is where probability density function live) functions are the entries ,addition ,multiplication by scalars have the obvious definitions , and we add an operation that multiplies functions.

For linear function spaces that the complete with respect to a norm.,the most important flavour of algebra is a Banach algebra with the following properties ,(o multiplaction operator ,which is undefined for the moment, λ is a scalar ,and $\| \ \|$ is the norm on the space)

(1) f o(g of)=(f0g)0h

(2) f o (g +h)=f0g+f0h

(3) (f +g)oh=f0h+goh

(4) λ (f o g)=(λ f)o g +f o(λ g)

(5) $\|fog\| \le \|f\|\|g\|$

Since $L^1(\Re)$ is not closed under ordinary multiplication of functions ,we need a different multiplication operation, and convolution is the most useful possibility.

To verify closure , if f, g $\in L^1(\Re)$,

$$\|fog\| = \int_{\Re} \left| \int_{\Re} f(y-x)g(x)dx \right| dx$$

$$\le \int_{\Re} \int_{\Re} |f(y-x)||g(x)|dxdy$$

$$= \int_{\Re} [\int_{\Re} |f(y-x)|dy]|g(x)|dx \text{ , by Fubini's theorem}$$

$$= \int_{\Re} [\int_{\Re} |f(z)|dz]|g(x)|dx$$

$$= \int_{\Re} \|f\||g(x)|dx = \|f\|\|g\|$$

$$\|fog\| = \|f\|\|g\|$$

This verifies the property (5),the norm condition, and is sometimes called Young's inequality., similarly we verify that $\|gof\| = \|g\|\|f\|$ .as well as the convolution algebra is commutative ;f*g=g*f.

For computing the p d f of a product of random variables ,the key results will be that the Mellin integral transform of a Mellin convolution is the product of Mellin integral transforms of the convolution functions.

$$M[fog] = \int_0^\infty [\int_0^\infty f(\frac{z}{w})g(w)\frac{dw}{w}]z^{s-1}dz$$

$$= \int_0^\infty [\int_0^\infty f(\frac{z}{w})z^{s-1}dz]g(w)\frac{dw}{w} \text{ , put } y=\frac{z}{w}, dy=\frac{dz}{w}, dz=wdy$$

$$= \int_0^\infty [\int_0^\infty f(y)(yw)^{s-1}wdy]g(w)\frac{dw}{w}$$

$$=\int_0^\infty\int_0^\infty f(y)y^{s-1}w^{s-1}g(w)dydw$$

$$=\int_0^\infty y^{s-1}f(y)dy\int_0^\infty w^{s-1}g(w)dw$$

=M[f](s)o M[g](s)

M[fog]=M[f](s)o M[g](s)

$$M[f0(g0h)](r,s,p)=\int_0^\infty\int_0^\infty\int_0^\infty x^{r-1}y^{s-1}z^{p-1}[f0(g0h)]dxdydz$$

$$=\int_0^\infty x^{r-1}fdx\int_0^\infty\int_0^\infty y^{s-1}z^{p-1}(g0h)dydz$$

$$=\int_0^\infty x^{r-1}fdx\int_0^\infty\int_0^\infty y^{s-1}z^{p-1}(g0h)dydz$$

=M[f}(s)M[g0h}(s,p)

$$M[f0(g0h)]=\int_0^\infty x^{r-1}y^{s-1}fogdx\int_0^\infty\int_0^\infty z^{p-1}hdz$$

M[f0(g0h)] =M[f0g}(r,s)M[h](p), then

M[f0(g0h)] (r,s,p)=M[f}(s)M[g0h}(s,p)=M[(fog)0h}(r,s,p)

M[f0(g0h)](r,s,p)=M[f0g](r,s)+M[f0h](r,p)

M[λ (fog)](r,s)=M[(λ f)og](r,s)+M[fo(λ g)](r,s)

Also the Mellin integral transform of Mellin convolution of $f_1......f_n$ is

M [$f_1......f_n$](s)=M[f $_1$](s).........M[f $_n$](s)

3.1.5.12: The Mellin Integral Transform and relation with Laplace Transform

If $f\in M_c(\Re)$ for all $c\in[a,b]$, we say that $f\in M_{[a,b]}(\Re)$,then we define Mellin integral transform of f with argument

$$F(s)=M\{f(u),s\}=\int_0^\infty u^{s-1}f(u)du \text{ ,where } a\le \mathrm{Re}(s)\le b$$

The inverse Mellin transform is

$$F(x)=M^{-1}[f](s)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}F(s)ds$$

The condition that the inverse exists is that $F(s)x^{-s}$ is analytic in a strip $(a,b)X(-\infty,\infty)$ such that $c\in[a,b]$ The mellin integral transform is derived from Laplace integral transforms follows

$$L[f(t),s]=\int_{-\infty}^\infty e^{-st}f(t)dt,$$

substitute x= e^{-t} , t=-log(x), dt= - $\frac{dx}{x}$,if t=- ∞ then x= ∞ and if t= ∞ then x=0

$$L\,[f\,(t)\,,\,s]= \int_{-\infty}^{\infty} (e^{-t})^s f(t)dt ,= \int_{\infty}^{0} x^s f(-\log x)(\frac{-dx}{x})$$

$$= \int_{0}^{\infty} x^{s-1} f(x)dx = M\,[f(x),s],$$

this is the Mellin integral transform of f(x) of the Mellin kernel x^{s-1} ,s>0 is the parameter.

The inverse Mellin integral transform is

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}F(s)ds$$, whenever this integral is exists.

The some technique is used to obtain the Mellin inversion theorem from the Laplace inverse

$$f(y)=T^{-1}[\overline{f}(.)](s) = \int_{-\infty}^{\infty} \overline{f}(s)e^{s\log y}ds$$

substitute s= $-(\eta - c)$, ds = $-d\eta$, limits are $c - i\infty$ to $c + i\infty$,then

$$f\,(y)=T^{-1}[\overline{f}(.)](y) = \int_{c-i\infty}^{c+i\infty} \overline{f}(-(\eta - c))e^{-(\eta-c)\log y}d\eta$$

$$= \int_{c-i\infty}^{c+i\infty} \overline{f}(-(\eta - c))y^{-\eta}y^{c}d\eta$$

$$= y^{c}\int_{c-i\infty}^{c+i\infty} \overline{f}(-\eta + c))y^{-\eta}d\eta$$

$$f\,(y)\,y^{-c} = \int_{c-i\infty}^{c+i\infty} \overline{f}(\eta)y^{-\eta}d\eta = f^{*}(\eta)$$

3.1.5.13: Product od Random Variablrs

Suppose we have random variables X ,Y with pdfs f_X , f_Y , and the product Z=XY determined, Consider the transformation $\psi : \Re^2 \to \Re^2$ given by $\psi(x, y) = (x, x^2 y) = (x, z)$ Expect at x=0, ψ is injective with (x,y)= $\psi^{-1}(x, z) = (x, \frac{y}{x^2})$ and the jacobian of ψ^{-1} is

$$J= \begin{vmatrix} \frac{\partial \psi_1^{-1}}{\partial x} & \frac{\partial \psi_2^{-1}}{\partial x} \\ \frac{\partial \psi_1^{-1}}{\partial y} & \frac{\partial \psi_2^{-1}}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & \frac{-2z}{x^3} \\ 0 & \frac{1}{x^2} \end{vmatrix} = \frac{1}{x^2}$$

Then using the multivariate change of variable theorem, the marginal density of Z is computed from the joint density of X and Y as

$$f_Z(z) = \int_{\Re} f_{XZ}(x,z)dx$$

$$= \int_{\Re} f_{XY}(\psi^{-1}(x,z))\frac{1}{x^2}dx$$

$$= \int_{\Re} f_{XY}(x,\frac{z}{x^2})\frac{1}{x^2}dx$$

$$= \int_{\Re} f_X(x) f_Y(\frac{z}{x^2})\frac{1}{x^2}dx,$$

by independent of X and Y

$$= f_X * f_Y(z)$$

This is precisely the Mellin convolution of f_X and f_Y . In principle , this plus the extensibility result (1) produces a way of finding product densities for arbitrary numbers of random variables.

3.1.5.14: Examples

As a simple illustration of the use of the Mellin transform , we use the belt and pully example .Recall that X-uniform (1.95,2.05) ,Y-uniform (1.45,1.55) and we seek the pdf of product XY.

The problem can be simplified by using the fact that a uniform (α, β) random variables can be expressed as $\alpha + (\beta - \alpha)U$, where U is uniform (0,1) random variable with p d f $I_{(0,1)}(x)$. In this case ,X=1.95+1U,Y=1/45+1U.Then XY=2.8275+34U+.01U ^{2}. Since we already know how to compute sums, the problem reduce to finding the pdf for the product of two uniform (0,1) random variables.

For Z=U ^{2}, the Mellin transformation evaluated to

$$f_Z(z) = \int_{\Re} f_X(x) f_Y(\frac{z}{x^2})\frac{1}{x^2}dx = \int_{\Re} \frac{1}{x^2}dx = [\frac{-1}{x}]_z^1 = -1 + \frac{1}{z}$$

$$= \frac{1}{z} - 1 \ , \ 0 < z \le 1$$

The bounds for the integration come from $x \le 1$ and $y \le 1 \Rightarrow x \ge z$

The result can also be obtained as $M^{-1}\{M[fu](s^2)\}(x)$, fu is the pdf of u.

We have M[fu](s)= $\int_0^1 [x^{s-1} - 1]dx = \frac{1}{s} - 1$, so we need

$$M^{-1}\left\{\frac{1}{s^2}\right\}(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{z^{-s}}{s^2}ds$$

In this simple case of the product of two uniform (0,1) RVs it is easier to compute the Mellin convolution directly, but the use of Mellin transforma allows computation of the pdf for the product of n uniform (0,1) RVs almost as early, yielding $\frac{(z^{-1})^{n-1}}{(n-1)!} - \frac{1}{(n-1)!}$

3.1.6. Remarks

1.Probability Background and Terminology for MIT is given

2.MIT and Contineous random Variable are defined

3.Probability Density Function for Contineous Random Variable is defined
4.Contineous Distribution Function is defined
5.Probabilities of Distribution Function F(x) of a CRV.
6. The derivatives of F(x) is defined
7.Expectation and Moments about origin
8.Moments about Mean,Varience,Skewness and Kurtisis
9.Measure of Skewness and Kurtosis
10.Mode, Median, Quartiles, Deciles, Percentiles, QD,CQD, Bowley's and Karl Pearson's Method for Coefficient of Skewness
11.Mean Deviation from Mean
12.MIT for sum of the Random variables
13.MIT for the product of Random variables
14.The MIT and relation with Laplace transform
15.Product of Random variables
16.Illustrated by Exanple

CONCLUSION

We have presented some background on statistics and probability theory and motivated to compute probability density functions for sum and multiplication of continuous random variables. The use of the Laplace transform to evaluate the convolution integral for the p d f of sum is relatively simple .The use of the Mellin integral transform to evaluate the convolution integral for the p d f of a product is known in the theory of integral transforms.

The use of the Laplce integral transform for some of the random variables is mostly used and explained in every advanced statistics text, now brief theory of Mellin integral transform for statistics and probability is given in this paper .It seems for any statisticians, mathematicians and engineers will also take interest in developing Mellin transform with statistics and probability.

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