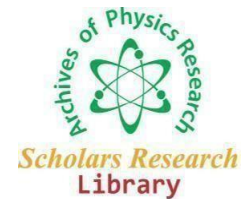




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The Nature of Quantum Theory James Moffat*

Department of Physics, University of Aberdeen, Aberdeen, United Kingdom

*Corresponding Author: James Moffat, Professor, Department of Physics, University of Aberdeen, Aberdeen, United Kingdom, E-mail: james.moffat@abdn.ac.uk

ABSTRACT

A discussion of the fundamental nature of non-relativistic and relativistic Quantum Theory, including Dirac spinors and the Dirac Operator.

Keywords: Fundamental, Quantum, Theory, Dirac

INTRODUCTION

Quantum Theory is a guess at how inputs to an experiment at the quantum level relate to outputs. These inputs and outputs are real numbers read on displays in the domain of Newtonian mechanics. The experiment can be real or conjectural, leading to a test of validation or predictive power. Good guesses are those which are consistent with all the existing experimental data (the facts) and have high predictive power, within their domain of application. This approach will help us to address issues such as 'don't ask questions just calculate' as we will see. Our currently accepted theory is the simplest of the good guesses, because the simplest explanation consistent with the facts is, we assume, nearest to the truth.

From these preliminary remarks we can assume that a good model of quantum theory will have the following characteristics:

1. It will have an elegant mathematical structure, since this is the simplest way of expressing its logic
2. It will be a theory based on the transformation of existing information into new information
3. It will be a black box model, with full access to its inputs and outputs but potentially incomplete information about why it works, since we do not require this of a good guess, model or theory. All we require is that for a given set of inputs, the outputs are correct, within its domain of application

WAVE MECHANICS ON HILBERT SPACE

For our purposes here, we define a free matter wave to be a disturbance which propagates with constant velocity in a given direction x . This disturbance is assumed to have a constant profile which does not decay over time. It follows that the profile of the matter wave at time t , $\psi(x(t), t)$, is the same as the profile at time zero; $\psi(x(0), 0)$, which equals $\psi(x(t)-ct, 0)$ where c is the velocity with which the disturbance moves. Setting $\tau=x(t)-ct$ leads to the profile function $G(\tau) = \psi(\tau, 0)$ and the general wave equation ;

$$\frac{\partial^2 \psi(x(t), t)}{\partial t^2} = c^2 \frac{\partial^2 G(\tau)}{\partial \tau^2}; \frac{\partial^2 \psi(x(t), t)}{\partial x^2} = \frac{\partial^2 G(\tau)}{\partial \tau^2}$$
$$\Rightarrow \frac{\partial^2 \psi(x(t), t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \psi(x(t), t)}{\partial t^2}$$

A periodic plane wave solution has the typical form

$$\psi(x(t), t) = \exp 2\pi i \left(\frac{x}{\lambda} - vt \right)$$

Where v is the frequency of this wave and λ is the wavelength. Substituting the de Broglie relations for the energy E and momentum p of a free particle;

$$E = h\nu; \quad p = \frac{h}{\lambda}$$

into this wave amplitude function gives;

$$\psi(x, t) = \exp \frac{2\pi i}{h} \left(\frac{hx}{\lambda} - hvt \right) = \exp \frac{2\pi i}{h} (px - Et)$$

This is the most basic example possible of a wave function. It corresponds to a free running particle moving in the x direction with energy E and momentum p . The set of all such wave functions for a single particle forms a space called a linear space or vector space. Given two wave functions $\psi_1(x, t); \psi_2(x, t)$ in this linear space we can also form the inner product;

$$\langle \psi_1(t) | \psi_2(t) \rangle = \int \psi_1^*(x, t) \psi_2(x, t) dx$$

The ability to form such inner products makes the linear space of wave functions into a Hilbert space. These wave functions relate to probabilities, as we will see later, so for a wave function state $\Psi(x, t)$ we also require that

$$\langle \psi(t) | \psi(t) \rangle = \int \psi^*(x, t) \psi(x, t) dx = 1$$

for all times t .

THE MEASUREMENT PROCESS

Now we describe the process of measuring this energy. Let's go back to our quantum wave function Ψ for a free particle moving in the direction x with energy E and momentum p . We first note that if we differentiate this expression in terms of time t then it pulls out the value E for the energy from the right-hand side, allowing it to be measured;

$$\begin{aligned} i \frac{\partial}{\partial t} \left\{ \exp \frac{2\pi i}{h} (px - Et) \right\} \\ = E \left\{ \exp \frac{2\pi i}{h} (px - Et) \right\} \\ = E \psi(x, t) \end{aligned}$$

In other words, the linear operator $i\hbar \frac{\partial}{\partial t}$ has E as an eigenvalue for the energy eigenvector $\psi(x, t)$. Similarly, the linear operator $-i\hbar \frac{\partial}{\partial x}$ has momentum eigenvalues p ;

$$-i\hbar \frac{\partial}{\partial x} \psi(x, t) = -i\hbar \frac{\partial}{\partial x} \left\{ \exp \frac{2\pi i}{h} (px - Et) \right\}$$

$$= p \left\{ \exp \frac{2\pi i}{h} (px - Et) \right\}$$

$$= p\psi(x, t)$$

Let $\hat{H} = i\hbar \frac{\partial}{\partial t}$ so that for our basic wavefunction ψ we have $\hat{H}\psi = E\psi$

Then;

1. \hat{H} is a self-adjoint linear operator since its eigenvalues is real numbers
2. Any reasonable wave function Ψ can be expressed as a weighted sum of these basic wavefunctions

Taking these two together means that if Ψ_j are the basic wavefunctions corresponding to energy measurements E_j (such as the energy levels of an electron in a hydrogen atom), and Ψ is a general wavefunction of the particle, then we can measure the energy of the quantum state defined by this general wavefunction as follows;

We have complex numbers α_j such that;

$$\psi = \sum_j \alpha_j \psi_j$$

And then the fact that the energy observable is a linear operator means that when we apply it to Ψ we get;

$$\hat{H}\psi = \sum_j \alpha_j \hat{H}\psi_j = \sum_j \alpha_j E_j \psi_j$$

Now instead of one energy level there is a weighted combination of several of them.

Because we insist that all wave function states, both the basic 'pure states' Ψ_j and the general one Ψ are normalized, this implies that

$$\sum |\alpha_j|^2 = \langle \psi, \psi \rangle = 1$$

We interpret the $|\alpha_j|^2$ as probabilities adding up to one which means that they cover all the options.

Measurement of the energy of a general quantum state is then interpreted as follows.

With probability $|\alpha_j|^2$ we project down onto the basic eigenfunction Ψ_j and then the measured value of the energy is the corresponding eigenvalue E_j .

THE HEISENBERG - DIRAC VIEWPOINT

From the point of view of linear operator theory (the Heisenberg – Dirac viewpoint), we can interpret this as follows:

1. A quantum state is an element $|x\rangle$ of a Hilbert space which is either finite or separable
2. The inner product $\langle x, y \rangle$ represents the (complex) amplitude of the transition from state $|x\rangle$ to $|y\rangle$
3. A system observable such as energy corresponds to a bounded linear operator acting on this state space
4. Energy measurements correspond to eigenvalues of this linear operator

If an ensemble of particles is measured, each prepared in the same initial energy state, then we will have a distribution of different energy measurements rather than a single value. Prediction of the nature of this distribution is a requirement of any theory. Only when the particle is in an eigenstate will the result be guaranteed. In quantum mechanics energy levels are spaced out or 'quantized' rather than smooth and continuous. We as humans don't see this discreteness directly because the spacing between energy levels is proportional to Planck's constant and this is a very small number. If we look at the energy emitted from a hydrogen atom through a spectrograph, however, we see these discrete levels. Each of these spectral lines corresponds to the energy emitted as a photon of light when the electron falls from a higher

energy level (energy eigenstate and eigenvalue) to a lower one. If the levels were not quantized they would form a continuous blur [1].

The measured value (eigenvalue) E is the quantum version of kinetic energy if the particle is free running. However, we want to apply this process to particles like the electron trapped inside a Hydrogen atom and most certainly not free. Hydrogen is the simplest and lightest element consisting of a positively charged proton trapping a single negatively charged electron. The separation distance is about a tenth of a nanometer—at this small scale quantum effects are dominant.

From this ‘picture’ we can see that our electron has both kinetic and potential energy, the potential energy deriving from its location relative to the proton in the middle. This potential energy is due to the electrostatic attraction between particles of opposite charge (the + of the proton and the – of the electron) and the strength of the attraction goes as $1/r^2$ as where r is the distance between the two particles—the smaller the distance the greater the force of attraction.

THE SCHRODINGER VIEWPOINT

In the classical, non-quantum world, the sum of the kinetic and potential energies for a particle is called the Hamiltonian, denoted H . For our free particle we have just kinetic energy. The particle is not trapped. The classical (non-quantum) version of the Hamiltonian in this case is $H(x,t) = \{\text{kinetic energy at location } x \text{ at time } t\}$. Thus the total energy of the particle at location x at time $t = \text{kinetic energy}$

$$E(x,t) = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{1}{2m}p(x,t)^2$$

Where the momentum is mv and v is the speed of the particle in the direction x . From this we have the classical Hamiltonian

$$H(x,t) = \frac{1}{2m}p(x,t)^2$$

We know that a quantum system has to look like the classical non-quantum system at a larger scale as we pull back from the detail. This is called the correspondence principle. We infer that the quantum version of the Hamiltonian must be of the form;

$$\hat{H}(x,t) = \frac{1}{2m}\hat{p}(x,t)^2$$

Substituting in the linear operators for total energy and momentum gives;

$$\hat{H} \rightarrow i\hbar \frac{\partial}{\partial t}; \hat{p} \rightarrow -i\hbar \frac{\partial}{\partial x}$$

$$\hat{H}\psi(x,t) = \frac{\hat{p}^2}{2m}\psi(x,t) \rightarrow i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\hbar^2 \frac{\partial^2}{\partial x^2}\psi(x,t)$$

This is Schrodinger’s equation for a free particle. In the general case we also have a potential function of the form $V(x)$, normally depending only on position, as for the Hydrogen atom.

The Hamiltonian in this case is then the sum of the kinetic and potential energies;

$$\begin{aligned} \hat{H}(x,t)\psi(x,t) &= \frac{1}{2m}\hat{p}(x,t)^2\psi(x,t) + V(x)\psi(x,t) \\ &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t) \end{aligned}$$

Substituting for \hat{H} gives the general Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t) + V(x)\psi(x,t)$$

and the corresponding eigenvalue equation is $\hat{H}\psi = E\psi$ in other words;

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t) + V(x)\psi(x,t) = E\psi(x,t)$$

To solve this equation, we assume that the location and time components of the wave function can be separated out. Thus this theory is inherently non-relativistic.

RELATIVISTIC WAVES

Maxwell's equations for the propagation of light waves through the vacuum of empty space can be written in the compact form;

$$\begin{aligned} \nabla \cdot E &= 4\pi\rho; \nabla \cdot H = 0 \\ \nabla \times E &= -\frac{1}{c} \frac{\partial H}{\partial t}; \nabla \times H = \frac{1}{c} \frac{\partial E}{\partial t} \end{aligned}$$

Where ∇ is the gradient operator; E and H are now the standard terms for the electric and magnetic field strength vectors and ρ is a measure of the charge density. This symmetry between E and H leads to the plane wave solutions;

$$\nabla^2 E \equiv \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}; \nabla^2 H = \frac{1}{c} \frac{\partial^2 H}{\partial t^2}$$

If we consider the special case of a plane wave propagating along the x -axis, with the E vector pointing in the y -direction, we have;

$$E = (E_x, E_y, E_z)$$

With $E_x = E_z = 0$ and

$$E_{y(t)} = E_{y(0)} \exp 2\pi i(vt - kx) = E_{y(0)} \exp 2\pi i v \left(t - \frac{x}{c} \right)$$

We have from Maxwell's equations that:

$$\begin{aligned} \frac{\partial H_x}{\partial t} &= -c(\nabla_y E_z - \nabla_z E_y) \\ \frac{\partial H_y}{\partial t} &= -c(\nabla_z E_x - \nabla_x E_z) \\ \frac{\partial H_z}{\partial t} &= -c(\nabla_x E_y - \nabla_y E_x) \end{aligned}$$

Most of these expressions disappear, leaving just;

$$\frac{\partial H_z}{\partial t} = -c \nabla_x E_y = c \frac{2\pi i v}{c} E_{y(t)} = 2\pi i v E_{y(0)} \exp 2\pi i v \left(t - \frac{x}{c} \right)$$

Thus the H vector has a non-zero z -component which varies with the same phase as the non-zero y -component of the E vector.

We now consider a plane periodic wave disturbance $F(x(t), y(t), z(t), t)$ propagating through free space in the direction given by the vector

r with direction cosines (l, m, n) . This means that $\frac{r}{|r|}$ is the three dimensional vector (le_x, me_y, ne_z) where e_x, e_y and e_z are unit vectors

along the three space axes and $l^2 + m^2 + n^2 = 1$.

In three dimensions the set $\{x, y, z; lx + my + nz = |r|\}$ now defines a plane orthogonal to the direction of propagation r . The vector k defined

by the components $\frac{v}{c}(l, m, n) = \frac{1}{\lambda}(l, m, n)$ is called the wavevector. For an arbitrary vector p drawn from the origin and ending in the point (x, y, z) of the plane we now have that $k \cdot p = \frac{v}{c}(lx + my + nz)$. The equation of the disturbance can thus be written as;

$$F(p(t), t) = \exp 2\pi i(k \cdot p \pm vt)$$

Where p is an arbitrary 3 vector connecting the origin to a point on the plane orthogonal to the direction of propagation. The wavevector k points in the direction of propagation.

We now assume as a convention, that our wave is of fixed wavelength (i.e. it is monochromatic light) and propagates in the positive z -direction. This means that the E and H vectors are confined to the plane defined by the x and y axes, normal to the propagation direction. A classical light wave which is polarized in the x -direction has a space-time dependent electric field oscillating in the x -direction which is the real part of the expression;

$$E = E_0 e_x \exp 2\pi i(kz - vt)$$

In the same way, we can consider a y -polarized light wave which is the real part of;

$$E = E_0 e_y \exp 2\pi i(kz - vt)$$

Circular polarization (i.e. circular rotation of the E vector in the x - y plane) is produced by linearly combining an x -polarised plane E wave with a y -polarised plane E wave 90 degrees out of phase. The equation for such a wave is thus of the form;

$$\begin{aligned} E &= E_0 \left(\frac{1}{\sqrt{2}} e_x \exp 2\pi i(kz - vt) + \left(\exp i \frac{\pi}{2} \right) \frac{1}{\sqrt{2}} e_y \exp 2\pi i(kz - vt) \right) \\ &= E_0 \left(\frac{1}{\sqrt{2}} e_x \exp 2\pi i(kz - vt) + \frac{i}{\sqrt{2}} e_y \exp 2\pi i(kz - vt) \right) \end{aligned}$$

These various classical linear combinations can more compactly be described by Jones vectors, which record the weights of the x and y components of E .

An x -direction direction polarized beam with weight $E_0 e_x$ thus corresponds to the Jones vector $\begin{pmatrix} E_0 e_x \\ 0 \end{pmatrix} = E_0 e_x \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ while a y -direction

polarized beam corresponds $\begin{pmatrix} 0 \\ E_0 e_y \end{pmatrix} = E_0 e_y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to and the circularly polarized beam above corresponds to the Jones vector

$\frac{1}{\sqrt{2}} \begin{pmatrix} E_0 e_x \\ i E_0 e_y \end{pmatrix}$ Normalizing these so that the total weight in each case is 1 leads to the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$.

THE QUANTUM PHOTON AND WIGNER THEORY

It is now possible to use the correspondence principle to argue that a polarized beam is composed of a stream of equivalently 'polarized' photon states defined by these normalized Jones vectors in a 2-dimensional vector space, giving rise to the well-known two-state qubit formulation of a photon. This is a good way of building an initial intuition. The basic quantum axioms tell us we need to define the 'observable' spin matrix S and its measured spin values correspond then to the eigenvalues of this matrix. How do we do this?

The answer was essentially supplied by Wigner who studied the group of rotations of space-time and their representation as groups of operators on vector spaces. It brings into play another property of light; its speed is independent of the source of the light; shining a torch out of a moving car does not make the photons go faster [2].

In two dimensions, let the vertical axis be time t and the horizontal axis be one-dimensional space x . A positive velocity is a vector pointing from the origin of these axes into the upper right quadrant to the point $(\Delta x, \Delta t)$.

Speed is distance over time (the vector pointing angle) so changes in the speed correspond to rotations of space-time. If c , the speed of light, is a finite universal constant then $c^2 (\Delta t)^2 - (\Delta x)^2 = 0 = (ic\Delta t)^2 + (\Delta x)^2$ in any coordinate system. Moving a line around and preserving its length s ; where $s^2 = (ic\Delta t)^2 + (\Delta x)^2$ is Euclidean geometry in space-time. Thus the constancy of the speed of light is equivalent to using coordinates ict and x and assuming space-time is Euclidean ('flat').

In this Euclidean space-time a change of speed is still just a rotation, so we have the coordinate transformation;

$$\begin{aligned} \begin{pmatrix} \Delta x' \\ ic\Delta t' \end{pmatrix} &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \Delta x \\ ic\Delta t \end{pmatrix} \\ \Rightarrow \Delta x' &= \cos \theta \Delta x - ic \sin \theta \Delta t = \cos \theta (\Delta x - ic \tan \theta \Delta t) \\ \Delta x' = 0 &\Rightarrow \Delta x = ic \tan \theta \Delta t \Rightarrow \tan \theta = \frac{V}{ic} \\ 1 = \cos^2 \theta + \sin^2 \theta &\Rightarrow \frac{1}{\cos^2 \theta} = 1 + \tan^2 \theta = 1 - \frac{V^2}{c^2} \\ \Rightarrow \Delta x' &= \cos \theta (\Delta x - ic \tan \theta \Delta t) = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \left(\Delta x - ic \left(\frac{V}{ic} \right) \Delta t \right) = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} (\Delta x - V \Delta t) \end{aligned}$$

Corresponding to the four vector $(c\Delta t, \Delta x, \Delta y, \Delta z)$ in space-time, there is the energy-momentum vector; composed of four dimensionally consistent components $\left(\frac{E}{c}, p_x, p_y, p_z \right)$

Consider now a massive particle of mass m , moving slowly with velocity $V \ll c$

Multiplying the dimensionally consistent space-time vector $(c\Delta t, \Delta x, \Delta y, \Delta z)$ by $\frac{m}{\Delta t}$ gives (mc, p_x, p_y, p_z) and we equate this to the energy-

momentum vector; $(mc, p_x, p_y, p_z) = \left(\frac{E}{c}, p_x, p_y, p_z \right)$

Since the energy-momentum vector is unique, the equivalence of mass and energy comes from equating the first element of each expression.

WIGNER LITTLE GROUPS

Returning to the massless case, If a photon is travelling with momentum p along the z -axis, then its total energy is $e=pc$ and the energy-momentum vector is then $(p, 0, 0, p)$. This is invariant only to the 'Wigner little group' of rotations in the x - y plane orthogonal to the direction of propagation (z). Rotations operating on this plane have the form;

$$\hat{R}(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \text{ With normalised eigenstates } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ and } \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

These eigenstates correspond to the Jones vectors for circular polarization of the classical EM wave, bringing the quantum and classical approaches into line. Relative to these eigenstates the spin \hat{S} is defined as $\hat{R}(\theta) = \exp\left(\frac{i}{\hbar} \hat{S}(\theta)\right)$ with eigenvalues ± 1 .

THE DIRAC EQUATION AND DIRAC OPERATOR

The Dirac equation describes the dynamics of the quantum state of a spin $\frac{1}{2}$ fermion (such as an electron) in a way which is consistent with the requirements of special relativity. This quantum state is four dimensional, with the first two components corresponding to the particle (e.g. the electron) and the second two components corresponding to the anti-particle (e.g. the positron). The Dirac spinor matrices γ^μ ($\mu=0,1,2,3$) are themselves 4x4 matrices. They satisfy the following ‘Clifford algebra’ relationship;

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \begin{cases} 0, & \mu \neq \nu \\ 2I, & \mu = \nu \end{cases}$$

where I is the 4x4 identity matrix (1s down the diagonal and 0s elsewhere)

In essence, by defining the operator;

$$\hat{D} = \sum_{j=0}^3 \gamma_j \frac{\partial}{\partial x_j}; \hat{D}^2 = \nabla^2$$

Dirac was able to take the square root of the relativistic relationship

$$E^2 = p^2 c^2 + m^2 c^4 \Leftrightarrow -\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} c^4 \Leftrightarrow \hat{D}^2 \psi = -m^2 c^4 \psi$$

with p the momentum of the particle. In quantum operator terms, this result in both positive and negative square roots, corresponding to the particle and its antiparticle. These Dirac spinor matrices automatically incorporate the spin of the particle, showing that spin is an inherently relativistic effect. In more detail, we take here the ‘Weyl representation’ of the Dirac matrices so that;

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

Where σ^μ ($\mu=0,1,2,3$) are the 2x2 Pauli spin matrices, $\bar{\sigma}^0 = \sigma^0$ and $\bar{\sigma}^\mu = -\sigma^\mu$ ($\mu=1,2,3$).

Given these relationships we can show directly that

$$i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} -I_{2 \times 2} & 0 \\ 0 & I_{2 \times 2} \end{pmatrix}$$

where $I_{2 \times 2}$ is the 2x2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

This operator representation acts on a 4 dimensional vector space (in fact a 4 dimensional Hilbert space). Each normalized element of this vector space is a quantum state Ψ .

Defining γ^5 to be the composite matrix operator $i\gamma^0 \gamma^1 \gamma^2 \gamma^3$ we can write this state Ψ in the form

$\psi = \frac{1}{2}(I_{4 \times 4} - \gamma^5)\psi + \frac{1}{2}(I_{4 \times 4} + \gamma^5)\psi$. We define these two terms as the left handed and right handed spinor

components of the state, thus we define;

$$\psi_L = \frac{1}{2}(I_{4 \times 4} - \gamma^5)\psi \quad \text{and} \quad \psi_R = \frac{1}{2}(I_{4 \times 4} + \gamma^5)\psi$$

So that $\psi = \psi_L + \psi_R$

Now from above we know that

$$(I_{4 \times 4} - \gamma^5) = \begin{pmatrix} I_{2 \times 2} & 0 \\ 0 & I_{2 \times 2} \end{pmatrix} - \begin{pmatrix} -I_{2 \times 2} & 0 \\ 0 & I_{2 \times 2} \end{pmatrix} = \begin{pmatrix} 2I_{2 \times 2} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{Similarly, } I_{4 \times 4} + \gamma^5 = \begin{pmatrix} 0 & 0 \\ 0 & 2I_{2 \times 2} \end{pmatrix}$$

Thus the four vector ψ_L has only two non-zero elements (the first two) and is of the form $\psi_L = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix}$. Similarly, the four vector ψ_R has

only two non-zero elements (the last two) and is defined as $\psi_R = \begin{pmatrix} 0 \\ 0 \\ -1 \\ \chi \\ -2 \\ \chi \end{pmatrix}$. These then correspond to the left handed 'chiral' Weyl spinor

$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and the right handed chiral Weyl spinor $\begin{pmatrix} -1 \\ \chi \\ -2 \\ \chi \end{pmatrix}$. In four dimensions we have $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ which is an element of a four

dimensional Hilbert space. The left handed Weyl spinor quantum state ψ_L corresponds to the particle and the right handed Weyl spinor quantum state ψ_R corresponds to the anti-particle. If ψ is a general Dirac spinor, it is a solution of the massless form of the Dirac equation:

$$\sum_{\mu} i\gamma^{\mu} (\partial_{\mu} - ieA_{\mu})\psi = 0$$

Where ψ has charge e and A_{μ} ($\mu=0,1,2,3$) is the vector potential corresponding to an external magnetic field. Note that the case with mass m non-zero simply introduces an additional term into the Dirac equation of the form $m\psi$ and the same argument goes through.

REFERENCES

- [1] Braddick, H.J., 1965. Vibrations, waves and diffraction. *McGraw-Hill Publishing Co*, USA.
- [2] Wigner, E., 1989. On unitary representations of the inhomogeneous lorentz group. *Nucl Phys B*, 6, pp. 9-64.