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The Role of Entropy in a Leaky Cavity and a comparative study between Einstein approach and Reservoir Theory

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Abstract

In the present work we shall work out the law of black body radiation by reservoir theory and make a comparison between rate equation approach of Einstein and general reservoir theory. We shall particularly discuss the role of entropy in our problem of interest.

Keywords: reservoir theory, stimulated emission entropy average photon number

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Introduction

The notion of stimulated emission was first advanced by Einstein [1] in 1917 and worked out the new spectral distribution formula of black body radiation discovered by Max Planck. Einstein's derivation was presumably different from Planck or Bose. In addition to these derivations, the law of distribution of black body radiation may also be worked out by the so-called reservoir theory [2],[3],[4].Historically Einstein's derivation of the radiation formula using the notion of stimulated emission and principle of detailed balancing is of considerable significance. It is seen that Einstein's attempt to rederive the radiation formula gave first to the concept of stimulated emission of far reaching consequence in the development of MASER and LASER. In the present work we make a comparative study between the rate equation approach of Einstein and the general reservoir theory. We shall particularly discuss the role of entropy in our problem of interest that is leaky cavity, as given in the work of Lang et al. [5].

The reservoir concept

In most of the areas of quantum optics, however we are interested in only part of the entire system as for example in laser system; we are interested to know the field but not interested what happens to atom. Atoms constitute the reservoir which is very much analogous to thermo dynamical reservoir. We can eliminate the reservoir by using reduced density operator. In our subject of study we have introduced the reservoir concept by considering a system consists of a simple harmonic oscillator interacting with a beam of two level atoms, some of which are initially in excited. The atomic energy distribution is characterized by a temperature T given by Boltzmann distribution

$$\frac{r_a}{r_b} = \exp(-\hbar\omega / k_B T), \qquad (1)$$

where r_a *and* r_b are number of atoms per second in the upper and lower levels which pass through the cavity. The effect of the beam on the oscillator is to bring it to the equilibrium temperature T.

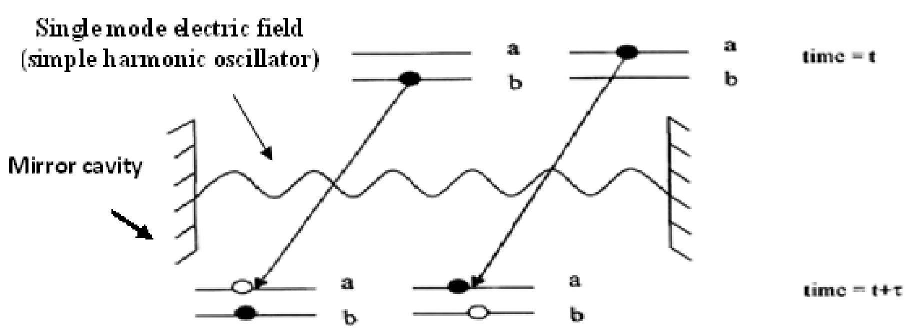

Fig I: A Cavity with two mirrors

(Two level atoms initially (at time-t) either in the upper state |a> or lower state |b>, Atoms initially in the upper state pass through per second and interact with the field; bringing it to an equilibrium temperature T. This is characteristics of the reservoir in the present case.)

In general, the field is described by a mixture of states and conveniently represented by the field density operator. We seek a coarse grained equation of motion for the laser radiation density matrix as it evolves due to addition of many excited atoms; equation is derived by using atom field density operator and by state vector for the atom coupled to a single state of the field.The time rate of change for $P(t)$ is $P(t) = r_a(\delta p)_a + r_b(\delta p)_b$ where $(\delta p)_a$ and $(\delta p)_b$ are the changes caused by the interaction with atoms injected in the upper and lower state respectively. We seek a coarse grained equation of motion for the laser radiation density matrix as it evolves due to addition of many excited atoms. We derive the equation by using atom field density operator $P_{a\text{-}t}(t)$ and by state vector. $|\psi_{atom}$- field (t)> for the atom coupled to a single state of the field.

The resulting atom field operator is given by

$$P_{a-t}(t+\tau) = \sum p_\psi \left|\psi_{a-f}(t+\tau)\right\rangle\left\langle\psi_{a-f}(t+\tau)\right| \qquad \ldots\ldots\ldots\ldots\ldots\ (1)$$

Where P_ψ is the probability of the field having the state vector $|\psi\rangle$representing the initial state vector at time (t) and atom field state vector at time $(t+\tau)$, coupling the atom field probability amplitudes in time according to equation of motion and taking the trace of its density matrix over the atomic states, we get the coarse grained time rate of change of density matrix element $\rho_{nm}(t)$ due to atoms initially in the lower state $|b>$ and similarly for atoms injected in to the state $|a>$.

The time rate of the change of density matrix element $\rho_{nm}(t)$ is given by

$$\dot{\rho}_{nm}(t) = \dot{\rho}_{nm}\Big|_{|a\rangle atoms} + \dot{\rho}_{nm}\Big|_{|b\rangle atoms} \qquad (2)$$

$$\dot{\rho}_{nm}(t) = -\frac{1}{2}\left[\Re_a(n+1+m+1) + \Re_b(n+m)\right]\rho_{nm} + \Re_a\sqrt{nm}\,\rho_{n-1,m-1} + \Re_b\left[(n+1)(m+1)\right]^{\frac{1}{2}}\rho_{n+1,m+1}$$

In particular, this gives the photon rate equation

$$\dot{\rho}_{nn}(t) = -\left[\Re_a(n+1) + \Re_b n\right]\rho_{nn} + \Re_a n\,\rho_{n-1,n-1} + \Re_b(n+1)\rho_{n+1,n+1} \qquad (3)$$

Where $\Re_a = r_a g^2 \tau^2$ single mode rate coefficient for state $|a>$

$\Re_b = r_b g^2 \tau^2$ Single mode rate coefficient for state $|b>$

g coupling constant in quantum theory of radiation

r_a and r_b excitation rates of states $|a>$ and $|b>$

Here each term is simply understood in the terms of probabilities that atoms do or don't make the transitions in the presence of a given numbers of photon. The equation can be understood in terms of Fig II.

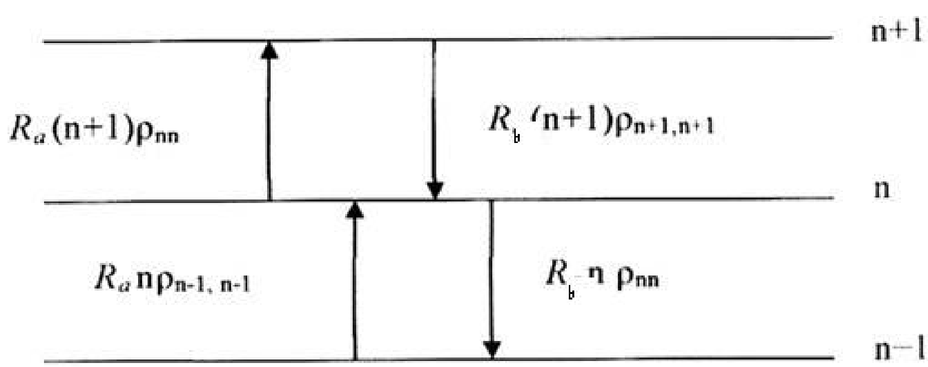

Fig. II. Diagram of photon number probability ρ_{nn}

With reference to Fig: II we would like to comment that there are three energy levels designated as n+1, n and n-1 the arrow pointing upward are termed as emission and the arrows pointing downward are indicated as absorption. This may give rise to some confusion because we find that the emission is always indicated as arrows pointing between two levels under normal convention. However we observe that the levels are designated in terms of photon numbers n and they are not energy levels but number states. The R_a term due to emission and R_b term is due to absorption.

The condition for arbitrary detailed balancing is

$$\dot{\rho}_{nn} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \ (4)$$

From equation (3) and (4)

$$\dot{\rho}_{nn} = -[\Re_a(n+1)+\Re_b n]\rho_{nn} + \Re_a n\,\rho_{n-1,n-1} + \Re_b(n+1)\rho_{n+1,n+1}$$
$$0 = -[\Re_a(n+1)+\Re_b n]\rho_{nn} + \Re_a n\,\rho_{n-1,n-1} + \Re_b(n+1)\rho_{n+1,n+1}$$
$$[\Re_a(n+1)+\Re_b n]\rho_{nn} = \Re_a n\,\rho_{n-1,n-1} + \Re_b(n+1)\rho_{n+1,n+1} \quad \ldots\ldots\ldots(5)$$

The Equilibrium is obtained in the system when the net flow all pairs of level vanishes. Assuming the reservoir of two level atoms in thermal equilibrium so $\Re_b\rho_{n+1,n+1} = \Re_a\rho_{nn}$ and $\Re_b\rho_{nn} = \Re_a\rho_{n-1,n-1}$

Replacing $\dfrac{\Re_a}{\Re_b}$ *by* $\dfrac{r_a}{r_b}$, we get $\rho_{nn} = \dfrac{r_a}{r_b}\rho_{n-1,n-1}$ and using the Boltzmann relation (1) we obtain $\rho_{nn} = \exp(-\hbar\omega/k_BT)\rho_{n-1,n-1}$ by .iteration of this result for successively lower n yields Planck's distribution for single mode $\rho_{nn} = \rho_{00}\exp(-\hbar\omega/k_BT)$

Again assuming the total probability $\sum_n \rho_{nn} = 1$ (normalization condition) we get $\sum_n \rho_{00}\exp(-n\hbar\omega/k_BT) = 1$

For simplification in calculation we have considered $\exp(-\hbar\omega/k_BT) = x$

We get

$$\rho_{00}\sum_n x^n = 1$$
$$\Rightarrow \rho_{00}(1+x+x^2+\ldots\ldots\ldots..) = 1$$
$$\Rightarrow \rho_{00}(1-x)^{-1} = 1$$
$$\Rightarrow \rho_{00} = 1-x$$

Thus from above we get $\rho_{00} = 1-\exp(-\hbar\omega/k_BT)$ and therefore the density matrix element $\rho_{nn} = [1-\exp(-\hbar\omega/k_BT)]\exp(-n\hbar\omega/k_BT)$ we can see that in the steady state the field comes to the temperature of the atomic beam reservoir as it should. Another quantity of interest is the average photon number, steady state value of this photon number can be determined. The expression for average photon number is given by

$$\langle n(t)\rangle = \sum n\rho_{nn}(t) = \sum n[1-\exp(-\hbar\omega/k_BT)\exp(-n\hbar\omega/k_BT)]\ldots\ldots\ldots\ldots(6)$$

Considering $x = \exp(-\hbar\omega/k_BT)$ we obtain the relation as

$$\langle n(t)\rangle = \sum_{n} n(1-x)x^{n} = \sum_{n=0} nx^{n} - \sum_{n=0} nx^{n+1}$$
$$\Rightarrow \langle n(t)\rangle = (x+2x^{2}+3x^{3}+..........)-(x^{2}+2x^{3}+3x^{4}+...........)$$
$$\Rightarrow \langle n(t)\rangle = x+x^{2}+x^{3}+..............$$
$$\Rightarrow \langle n(t)\rangle = (1+x+x^{2}+x^{3}+......)-1$$
$$\Rightarrow \langle n(t)\rangle = (1-x)^{-1}-1$$
$$\Rightarrow \langle n(t)\rangle = \frac{x}{1-x}$$
$$\Rightarrow \langle n((t)\rangle = \frac{y^{-1}}{1-y^{-1}} \quad considering \quad x = y^{-1}$$
$$\Rightarrow \langle n(t)\rangle = \frac{1}{y-1}$$

Thus putting $y = \exp(n\hbar\omega / k_{B}T)$ we get the average photon number

$$\langle n(t)\rangle = \sum_{n} n\rho_{nn}(t) = \frac{1}{\exp(\hbar\omega / k_{B}T)-1} \qquad ....\quad ..(7)$$

The steady state value of this number is given by the Bose-Einstein expression

$$and \quad \langle n(\alpha)\rangle = \sum_{n} n\rho_{nn} = \frac{1}{\exp(\hbar\omega / k_{B}T)-1} \qquad ....(8)$$

Rate equation and Einstein's relations:

It may be noted that in the Einstein's derivations of radiation formula a cavity was assumed with closed walls in thermal equilibrium. In which there are photons of energy $\hbar\omega$, N_n atoms in the ground state E_n & N_m atoms in the excited state E_m. The density of the photon is specified as ρ_{ω} .When an atom has suitable frequency of light falling on it, it can absorb that photon of light and make transition from m to n.The transition from lower to upper energy state occur in only due to absorption and transition from upper to lower occur by two alternative i.e. stimulated emission and spontaneous emission. The rate equation of change N_n atoms in the ground state is

$$\frac{dN_{n}}{dt} = -N_{m}A_{mn} - N_{m}B_{mn}\rho(\omega) + N_{n}B_{nm}$$

At thermal equilibrium $\frac{dN_{n}}{dt} = 0$ i.e. principle of detailed balancing then the rate equation becomes as

$$N_m B_{mn}\rho(\omega) + N_m A_{nm} = N_n B_{nm}\rho(\omega) \ldots\ldots\ldots\ldots\ (9)$$

Where N_m atoms in the ground state, N_n atoms in the excited state, B_{mn} probability of the simulated emission, B_{nm} probability of simulated absorption and ρ_ω density of photons. The basic assumptions of Einstein were $E_1 - E_2 = \hbar\omega$ and $N_m = N_n \exp(-n\hbar\omega / k_B T)$ now using the principles of detailed balancing Einstein writes, for any temperature,

$$\rho(\omega) = \frac{A_{nm}}{B_{nm}\exp(\hbar\omega / k_B T) - B_{mn}}$$

It is observed that the term $N_m A_{mn}$ does not change with temperature that the term involving ρ_ω increases with temperature, at high enough temperature N_m,A_{nm} can be neglected and also N_m=N_n so from (9) we get B_{mn}=B_{nm}=B.

$$\text{Therefore} \quad \rho(\omega) = \frac{A}{B}\frac{1}{\exp(\hbar\omega / k_B T) - 1}$$

This gives Planks formula [6] for the energy density radiation at thermal equilibrium provided the ratio of spontaneous and stimulated coefficient $\frac{A}{B} = \frac{h\omega^3}{\pi^2 c^3}$.

Comparison between reservoir and Einstein approach:

It is now worthwhile to make a comparative study between the two approaches as indicated in the earlier sections. In Einstein approach we have written the radiation formula as

$$\rho(\omega) = \frac{A}{B}\frac{1}{\exp(\hbar\omega / k_B T) - 1}$$

Where A/B =Ratio of coefficient of spontaneous and stimulated emission. And in reservoir approach a photon number distribution is given by

$$\langle n(t)\rangle = \sum_n n\rho_{nn}(t) = \frac{1}{\exp(\hbar\omega / k_B T) - 1}$$

Here presumably A/B= 1 this is a limiting case. We farther note that in Einstein's approach an enclosed cavity is assumed and there is no indication how laser action takes place but in reservoir theory using density operator method a FabryPerot cavity is already assumed where perfectly reflecting dielectric mirrors are used to amplify stimulated emission. It will be now of sufficient interest to find the role of entropy in the both the cases. In thermodynamic language reservoirs is defined as a body of such a large mass that it may absorb or reject an unlimited quantity of heat without experiencing an appreciable change in any other thermodynamic coordinate but it is to be noted that there is a small change in the reservoir when a finite amount of heat flows in and out of the reservoir but an extremely small one, too small to be measured, as for example, in the expansion of average photon number,

$$\langle n \rangle = \frac{1}{\frac{\hbar\omega}{k_B \mathrm{T}} + \frac{\hbar^2\omega^2}{2!\,k^2{}_B \mathrm{T}^2} + \ldots\ldots\ldots \quad \ldots}$$

As temperature increases heat flow in an out of the reservoir also increases but too small to measure, and this is also the direction to which entropy increases. The average photon numbers will considerable only when $\frac{\hbar\omega}{k_B \mathrm{T}} << 1$ this implies $\hbar\omega << k_B \mathrm{T}$ again interestingly for $\frac{\hbar\omega}{k_B \mathrm{T}} = 0$ the average photon number would be infinite, it may happen when T is infinite when T=0 than $\frac{\hbar\omega}{k_B \mathrm{T}} = \alpha$ (infinite) and average photon number will be zero, from here we may considers that average photon number increases when T increases. In the laser cavity, there is leakage of radiation through the mirror and eventually a radiation is bound to die down. In the case of Einstein cavity this information is not available, and in Einstein cavity the reservoir is outside the world. In this case as temperature increases a stage reached where the second term in the equation (9) is neglected at high temperature and is at maximum entropy. In equation (5) $\Re_a(n+1)\rho_{nn}$ *and* $\Re_b(n+1)\rho_{n+1,n+1}$ are ignored (balanced). This also happens at high temperature .Thus we come to a stage where we make fruitful analogy of two approaches. In Einstein approach the direction of entropy and direction of time are same as in the reservoir approaches, the direction of entropy and direction of time are in same direction.

According to the second law of thermodynamics entropy of a self controlled and closed system always increases in any natural processes. It is related to heat added to a system and its temperature; it is a statistical measure of the disorder of the system. It may be note that a cavity as indicated in figure (I) is always leaky due to the facts that mirrors are not perfectly reflecting. It is demonstrated by Lang et al (3) that leakage leads to a damping of the free oscillation in the laser cavity. At this we may also consider the cavity picture in terms of thermodynamic considerations a heat engine and Carnot cycle. Consider a four level system of laser as shown in Fig III.

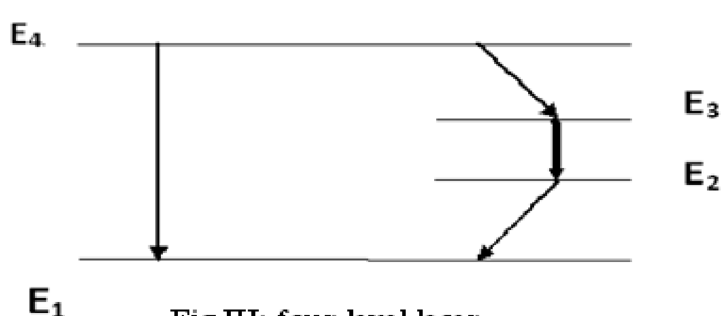

Fig III: four level laser

Pumping light induces transition between E_1 and E_4.takes place between E_4 and E_3.Laser action basically takes place between E_3 and E_2. This is a general principle that in atomic system where the lower level decays very fast by radiation to a set of levels and where the upper level is optically connected to ground level is ideally suited for laser excitation.

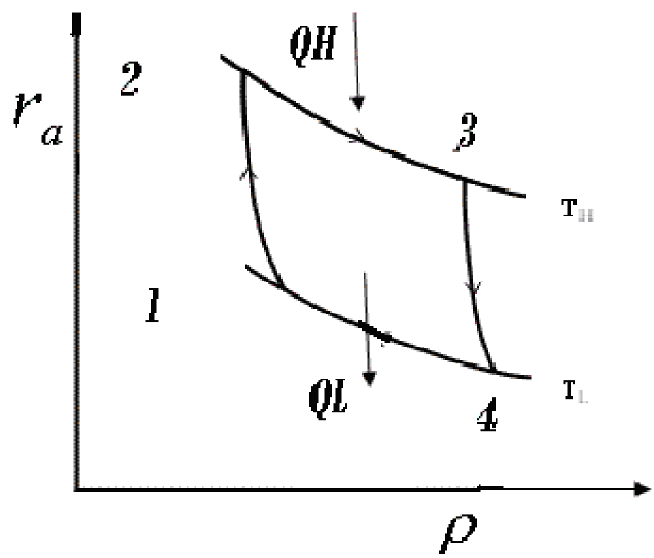

Fig IV : A Comparison between Carnot cycle and four level laser

Fig IV shows a diagram which was an analogy with the Carnot cycle of a real gas. This depicts the r_a- ρ diagram. The atoms are initially in this stage are representation by point 1, the four process are then

Processes 1-2 reversible adiabatic excitations until the temperature increases to $T_{H.}$
Processes 2-3 irreversible isothermal relaxation until any desired point 3 is reached.
Processes 3-4 irreversible adiabatic until the temperature $T_L$is obtained.
Processes 4-1 reversible isothermal relaxation until original state is reached.

It may be noted that in our considerations we have two reversible and two irreversible processes and total rate change of entropy is zero.

Conclusion

In the present work we have made a comparison between the reservoir theory and Einstein's approach for the derivation of Planks Black body radiation formula. It may be noted that Einstein approach is simplest manifestation of a system in equilibrium and the concept of stimulated emission. Principle of detailed balancing is used in both the approaches. We have also indicated how a four level laser may be viewed from a thermo dynamical point of view. This may be of great importance in near future in the development of laser physics and may help in understanding the mechanism of leaser physics for very short pulse laser.

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