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The use of linear parametric approximation in numerical solving of nonlinear non-smooth Fuzzy equations

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ABSTRACT

In this approach (The linear parametric approximation), the nonlinear functions is approximated by a piecewise linear functions. The obtained solution has desirable accuracy and the error is completely controllable. With extension this approach, we propose a new two-step iterative method for solving nonlinear fuzzy equations and nonlinear non-smooth fuzzy equations. Finally some numerical examples are given to show the efficiency of the proposed approach to solve same equations in the other references.

Key words: Taylor linear expansion, Linear Parametric Approximation, nonlinear non-smooth fuzzy function.

INTRODUCTION

In recent years much attention has been given to develop iterative type methods for solving nonlinear equations like F(x) = 0. Because the Systems of simultaneous nonlinear equation play a major role in various areas such as mathematics, statistics, engineering and social sciences. The concept of fuzzy numbers and arithmetic operation with these numbers were first introduced and investigated by [5,8,13–15,17]. One of the major applications of fuzzy number arithmetic is nonlinear equations whose parameters are all or partially represented by fuzzy numbers [1, 6, 10]. Standard analytical techniques presented by Buckley and Qu in [2 − 5]. Standard analytical techniques like Buckley and Qu method [1–4], cannot be suitable for solving the equations such as:

$$(i) ax^5 + bx^4 + cx^3 + dx - e = f,$$
$$(ii) x - \sin(x) = g,$$
$$(iii) f(x) = |x|,$$

Where x, a, b, c, d, e, f and g are fuzzy numbers. Moreover famous of classical numerical methods such as: Newton and Newton-Raphson are unable to solve the non-smooth equations such as: (iii) equation.We therefore need to develop the numerical methods to find the roots of such equations. Here, we consider these equations, in general, as: $F(x) = 0$.

In this paper we introduce a new approach to solve approximately nonlinear non-smooth fuzzy equations which don't have any limitation upon convexity and smoothness of the nonlinear fuzzy functions. In this approach any given nonlinear fuzzy function is approximated by a piecewise linear function with controlled error which is based on generalization of Taylor linear expansion of smooth function. Also we represent an efficient algorithm to solve of approximated fuzzy problem. One of the main advantages of our approach is that it can be extended to problems with nonlinear non-smooth fuzzy functions by introducing a novel definition of Global Weak Differentiation in the sense of L1-norm [19]. The paper is organized as follow:

In Section two, we recall some fundamental results of fuzzy numbers. In section three we explain the approach of linear parametric approximation for nonlinear equations. We verify in the fourth section the approach extended for non-smooth nonlinear equations by introducing the definition of global weak differentiation. We extended the approach in section five for solving fuzzy nonlinear equations. In the sixth sections, the approach was extended for solving of non-smooth nonlinear fuzzy equations. Finally some illustrative examples and conclusions are given to show the effectiveness of the proposed approach.

2. Preliminaries

Definition 2.1. A fuzzy number is a fuzzy set like $u:\Re \to I=[0,1]$ which satisfies [9,16,18],

1. u is upper semi continuous,
2. $u(x)=0$ outside some interval $[c,d]$,
3. There are real numbers a,b such that $c\le a\le b\le d$ and

3.1 $u(x)$ is monotonic increasing on $[c,a]$,

3.2 $u(x)$ is monotonic decreasing on $[b,d]$,

3.3 $u(x)=1, a\le x\le b$.

The set of all these fuzzy numbers is denoted by E. An equivalent parametric is also given in [20] as follows.

Definition 2.2. A fuzzy number u in parametric form is a pair $(\underline{u},\overline{u})$ of function $\underline{u}(r),\overline{u}(r), 0\le r\le 1$, which satisfies the following requirements:

1. $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
2. $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,
3. $\underline{u}(r)\le\overline{u}(r),\ 0\le r\le 1$.

A popular fuzzy number is the trapezoidal fuzzy number $u=(x_0,y_0,\sigma,\beta)$ with interval defuzzifier $[x_0,y_0]$ and left fuzziness σ and right fuzziness β where the membership function is:

$$u(x)=\begin{cases}\frac{1}{\sigma}(x-x_0+\sigma) & x_0-\sigma\le x\le x_0,\\ 1 & x\in[x_0,y_0],\\ \frac{1}{\beta}(y_0-x+\beta) & y_0\le x\le y_0+\beta,\\ 0 & \text{otherwise.}\end{cases}$$

Its parametric form is:

$$\underline{u}(r)=x_0-\sigma+\sigma r,\quad \overline{u}(r)=y_0+\beta-\beta r.$$

Let $TF(\Re)$ be the set of all trapezoidal fuzzy numbers. The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows.

For arbitrary $u=(\underline{u},\overline{u}), v=(\underline{v},\overline{v})$ and $k>0$ we define addition $(u+v)$ and multiplication by scaler k as:

$$(\underline{u+v})(r) = \underline{u} + \underline{v}, \quad (\overline{u+v})(r) = \overline{u} + \overline{v},$$
$$(\underline{ku})(r) = k\underline{u}(\mathrm{r}), \quad (\overline{ku})(r) = k\overline{u}(r).$$

3. The approach of linear parametric approximation for nonlinear equations [19]
Consider the $f(x)=0$ is a nonlinear smooth function. We may approximate the nonlinear function $f(x)$ by a piecewise linear function defined on [a, b]. Let us mention the following definitions.

Definition 3.1. Let $P_n([a,b])$ be a partition of the interval $[a,b]$ as the form:

$$P_n([a,b]) = \{a = x_0, x_1, \ldots, x_n = b\}$$

Where $h = \dfrac{b-a}{n}$ and $x_i = x_0 + ih$. The norm of partition defined by:

$$\|P_n([a,b])\| = \max_{1 \le i \le n} \{x_i - x_{i-1}\}$$

It is easy to show that $\|P_n([a,b])\| \to 0$ as $n \to \infty$.

Definition 3.2. The function $f_i(x, s_i)$ is defined as follows:

$$f_i(x, s_i) \overset{\Delta}{=} f'(s_i)x + f(s_i) - s_i f'(s_i); \;\; x \in [x_{i-1}, x_i] \qquad i = 1, \ldots, n$$

where $s_i \in (x_{i-1}, x_i)$ is an arbitrary point. The function $f_i(x\ , s_i)$ is called the linear parametric approximation of $f(x)$ on $[x_{i-1}, x_i]$ at the point $s_i \in (x_{i-1}, x_i)$. (In usual linear expansion the point s_i is fixed, but here we assume s_i is a free point in $[x_{i-1}, x_i]$.

Now, we define $g_n(x)$ as the parametric linear approximation of $f(x)$ on $[a,b]$ associated with the partition P_n as follows:

$$g_n(x) = \sum_{i=1}^{n} [f_i(x, s_i)\chi_{[x_{i-1}, x_i](x)}] \tag{1}$$

where χ_A is the characteristic function and defined as below:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

The following theorems are shown that $g_n(x)$ is convergence uniformly to the original nonlinear function $f(x)$ when $\|P_n([a,b])\| \to 0$. In the other word we show that:

$$g_n \to f \quad \text{unformaly on } [a,b] \;\; \text{as } \|P_n([a,b])\| \to 0$$

The following theorems are shown that $g_n(x)$ is convergence uniformly to the original nonlinear function $f(x)$ when $\|P_n([a,b])\| \to 0.$ in the other word we show that:

$$g \to f \;\; \text{unformly on } [a,b] \;\text{ as } \|P_n([a,b])\| \to 0$$

Lemma 3.1. Let $P_n([a,b])$ be an arbitrary regular partition of $[a,b]$. If $f(x)$ is continuous function on $[a,b]$ and $x, s \in [x_{i-1}, x_i]$ are an arbitrary points then:

$$\lim_{\|P_n([a,b])\| \to 0} f_i(x, s_i) = f(x_i).$$

Proof. The proof is an immediate consequence of the definition.

This lemma shown that $g_n \to f$ point-wise on $[a,b]$.

Definition 3.3. A family F of complex functions f defined on a set A in a metric space X, is said to be equicontinuous on A if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $d(x,y) < \delta, x \in A, y \in A, f \in F$. Here $d(x,y)$ denotes the metric of A (see [7]).

Since $\{g_n(x)\}$ is a sequence $|f(x) - f(y)| < \varepsilon$ of linear functions it is trivial that this sequence is equicontinuous.

Theorem 3.1. Let $\{f_n\}$ is an equicontinuous sequence of function on a compact set A and $\{f_n\}$ converges point-wise on A. Then $\{f_n\}$ converges uniformly on A.

Proof. See [19].

Theorem 3.2. Let $g_n(x)$ is a piecewise linear approximation of $f(x)$ on $[a,b]$ as (1). Then:

$$g_n \to f \ \textit{unformaly on} [a,b].$$

Proof. The proof is an immediate consequence of Lemma 3.1 and Theorem 3.1 in [19].

Now, we introduce a novel definition of global error for approximated $f(x)$ with linear parametric function $g_n(x)$ in the sense of L1-norm which is a suitable criterion to show the goodness of fitting.

Definition 3.4. Let $f(x)$ be a nonlinear smooth function defined on $[a,b]$ and let $g_n(x)$ defined in (4) be a parametric linear approximation of $f(x)$ .Let the global error for approximation of the function $f(x)$ with function $g_n(x)$ in the sense of L_1-norm is defined as follows:

$$E_n = \int_a^b |f(x) - g_n(x)| dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(x) - f_i(x)| dx$$

It is easy to show that E_n tends to zero uniformly when $\|P_n([a,b])\| \to 0$. This definition is used to make the fine partition which is matched with a desirable accuracy.

4. Extension of linear parametric approximation for solving fuzzy nonlinear equations

Now our aim is to obtain a solution for fuzzy nonlinear equation $\widetilde{F}(x) = 0$. The parametric form of two step method is as follows:

$$\begin{cases} \underline{F}(\underline{x}, \overline{x}, r) = 0, \\ \overline{F}(\underline{x}, \overline{x}, r) = 0, \end{cases} \quad \forall r \in [0,1] \tag{2}$$

Therefore, by use the linear parametric approximation approach (generalization of Taylor linear expansion of smooth function) of $\underline{F}$, $\overline{F}$, then $\forall r \in [0,1]$,we define the function $\tilde{F}_i(x, s_i, r) = 0$ as follows:

$$\begin{cases} \underline{F}_i(\underline{x}, \overline{x}, \underline{s}_i, r) \overset{\Delta}{=} \underline{f}'(\underline{s}_i, r)\underline{x} + \underline{f}(\underline{s}_i, r) - \underline{s}_i \underline{f}'(\underline{s}_i, r); \ \underline{x} \in [\underline{x}_{i-1}, \underline{x}_i] \\ \overline{F}_i(\underline{x}, \overline{x}, \overline{s}_i, r) \overset{\Delta}{=} \overline{f}'(\overline{s}_i, r)\overline{x} + \overline{f}(\overline{s}_i, r) - \overline{s}_i \overline{f}'(\overline{s}_i, r); \ \overline{x} \in [\overline{x}_{i-1}, \overline{x}_i] \end{cases} \quad i = 1, \dots, n \tag{3}$$

where $\underline{s}_i \in (\underline{x}_{i-1}, \underline{x}_i)$ and $\overline{s}_i \in (\overline{x}_{i-1}, \overline{x}_i)$ are an arbitrary point. The function $\underline{F}_i(\underline{x}_i, \underline{s}_i, r)$ is called the lower bound linear parametric approximation of $\tilde{F}_i(x, s_i, r)$ on $[\underline{x}_{i-1}, \underline{x}_i]$ at the point $\underline{s}_i$ and $\overline{F}_i(\overline{x}_i, \overline{s}_i, r)$ is called the upper bound linear parametric approximation of $\tilde{F}_i(x, s_i, r)$ on $[\overline{x}_{i-1}, \overline{x}_i]$ at the point $\overline{s}_i$.

Now, we define $\tilde{G}_n(x, s_i, r)$ as the parametric linear approximation of $\tilde{F}_i(x, s_i, r)$ on $[a, b]$ associated with the partition P_n as follows:

$$\begin{cases} \underline{G}_n(\underline{x}, \underline{s}_i, r) = \sum_{i=1}^{n} [\underline{F}_i(\underline{x}, \underline{s}_i, r) \chi_{\underline{A}}(\underline{x})] \\ \overline{G}_n(\overline{x}, \overline{s}_i, r) = \sum_{i=1}^{n} [\overline{F}_i(\overline{x}, \overline{s}_i, r) \chi_{\overline{A}}(\overline{x})] \end{cases} \tag{4}$$

where $\underline{A} = [\underline{x}_{i-1}, \underline{x}_i], \overline{A} = [\overline{x}_{i-1}, \overline{x}_i]$ and $\chi_{\underline{A}}$, $\chi_{\overline{A}}$ are the lower bound and upper bound characteristic functions respectively and defined as below:

$$\chi_{\underline{A}}(\underline{x}) = \begin{cases} 1 & \underline{x} \in \underline{A} \\ 0 & \underline{x} \notin \underline{A} \end{cases} \quad , \quad \chi_{\overline{A}}(\overline{x}) = \begin{cases} 1 & \overline{x} \in \overline{A} \\ 0 & \overline{x} \notin \overline{A} \end{cases} \tag{5}$$

The following theorems are shown that $\tilde{G}_n(x, s_i, r)$ is convergence uniformly to the original nonlinear fuzzy equation $\tilde{F}_i(x, s_i, r)$ when $\|P_n([a,b])\| \to 0$. In the other word we show that:

$$\begin{cases} \underline{G}(\underline{x}, \underline{s}_i, r) \to \underline{F}_i(\underline{x}, \underline{s}_i, r) \\ \overline{G}_n(\overline{x}, \overline{s}_i, r) \to \overline{F}_i(\overline{x}, \overline{s}_i, r) \end{cases} \text{unformly on } [a,b] \text{ as } \|P_n([a,b])\| \to 0$$

Lemma 4.1. Let $P_n([a,b])$ be an arbitrary regular partition of $[a,b]$. If $\overline{F}_i(\overline{x}, \overline{s}_i, r)$ and $\underline{F}_i(\underline{x}, \underline{s}_i, r)$ are continuous function on $[a,b]$ and $\underline{x}$, $\overline{x}$, $\underline{s}_i \in [\underline{x}_{i-1}, \underline{x}_i]$, $\overline{s}_i \in [\overline{x}_{i-1}, \overline{x}_i]$ are an arbitrary points then:

$$\lim_{\|P_n([a,b])\| \to 0} \Rightarrow \begin{cases} \underline{F}_i(\underline{x}, \underline{s}_i, r) = \underline{F}(\underline{x}_i, r) \\ \overline{F}_i(\overline{x}, \overline{s}_i, r) = \overline{F}(\overline{x}_i, r). \end{cases} \tag{6}$$

Proof. The proof is an immediate consequence of the definition. This lemma shown that

$\tilde{G}_n(x, s_i, r) \to \tilde{F}_i(x, s_i, r)$ point-wise on $[a, b]$.

Base on definition 2.4 and extension of it, $\underline{G}_n(\underline{x},\underline{s}_i,r)$ and $\overline{G}_n(\overline{x},\overline{s}_i,r)$ are sequence of linear fuzzy functions it is trivial that this sequence is equicontinuous. Moreover base on Theorem 3.1, $\underline{F}_i(\underline{x},\underline{s}_i,r)$ and $\overline{F}_i(\overline{x},\overline{s}_i,r)$ converges uniformly on $\underline{A},\overline{A}$ in metric spaces $\underline{X}$, $\overline{X}$ respectively.

Theorem 4.1. Let $\{\underline{F}_n\}$and $\{\overline{F}_n\}$ are equicontinuous sequence of function on a compact set s of $\underline{A},\overline{A}$ respectively and $\{\underline{F}_n\}$, $\{\overline{F}_n\}$ converges point-wise on $\underline{A},\overline{A}$. Then $\{\underline{F}_n\}$, $\{\overline{F}_n\}$ converges uniformly on $\underline{A},\overline{A}$ respectively.

Proof. Since $\{\underline{F}_n\}$, $\{\overline{F}_n\}$ are sequences of equicontinuous fuzzy function on $\underline{A},\overline{A}$ then:

$$\forall\ \varepsilon>0\ \exists\ \delta>0\ s.t$$

$$\begin{cases} d(\underline{x},\underline{y})<\delta \\ d(\overline{x},\overline{y})<\delta \end{cases} \rightarrow \begin{cases} \left|\underline{f}_n(\underline{x})-\underline{f}_n(\underline{y})\right|<\varepsilon \\ \left|\overline{f}_n(\overline{x})-\overline{f}_n(\overline{y})\right|<\varepsilon \end{cases} \underline{x},\underline{y}\in\underline{A};\quad \overline{x},\overline{y}\in\overline{A};\quad n=1,2,\cdots.$$

For each $\underline{x}\in\underline{A},\overline{x}\in\overline{A}$ there exists $\delta>0$ such that $\underline{A}\subseteq\bigcup_{\underline{x}\in\underline{A}}N(\underline{x},\delta)$, $\overline{A}\subseteq\bigcup_{\overline{x}\in\overline{A}}N(\overline{x},\delta)$ .Since $\underline{A},\overline{A}$ are compact, this open covering of $\underline{A},\overline{A}$ have a finite sub-covering. Thus, there exists a finite number of points such as: $\underline{x}_1,\underline{x}_2,\ldots,\underline{x}_r$ in $\underline{A}$ and $\overline{x}_1,\overline{x}_2,\ldots,\overline{x}_r$ in $\overline{A}$ such that $\underline{A}\subseteq\bigcup_{i=1}^{r}N(\underline{x}_i,\delta)$, $\overline{A}\subseteq\bigcup_{i=1}^{r}N(\overline{x}_i,\delta)$ .Therefore for each $\underline{x}\in\underline{A}$ and $\overline{x}\in\overline{A}$ there exists $\underline{x}_i\in\underline{A},\overline{x}_i\in\overline{A}$ respectively for $i=1,2,\ldots,r$; such that:

$$\begin{cases} d(\underline{x},\underline{x}_i)<\delta \\ d(\overline{x},\overline{x}_i)<\delta \end{cases}$$

We know $\underline{F}_n,\overline{F}_n$ are point-wise convergent sequences then there exists a natural number N such that for each $n\geq N, m\geq N$ we have:

$$\begin{aligned}\left|\underline{F}_n(\underline{x})-\underline{F}_m(\underline{x})\right| &= \left|\underline{F}_m(\underline{x})-\underline{F}_m(\underline{x}_i)+\underline{F}_m(\underline{x}_i)-\underline{F}_n(\underline{x}_i)+\underline{F}_n(\underline{x}_i)-\underline{F}_n(\underline{x}\ )\right| \\ &\leq \left|\underline{F}_m(\underline{x})-\underline{F}_m(\underline{x}_i)\right|+\left|\underline{F}_m(\underline{x}_i)-\underline{F}_n(\underline{x}_i)\right|+\left|\underline{F}_n(\underline{x}_i)-\underline{F}_n(\underline{x}\ )\right| \leq 3\varepsilon.\end{aligned} \tag{7}$$

$$\begin{aligned}\left|\overline{F}_n(\overline{x})-\overline{F}_m(\overline{x})\right| &= \left|\overline{F}_m(\overline{x})-\overline{F}_m(\overline{x}_i)+\overline{F}_m(\overline{x}_i)-\overline{F}_n(\overline{x}_i)+\overline{F}_n(\overline{x}_i)-\overline{F}_n(\overline{x}\ )\right| \\ &\leq \left|\overline{F}_m(\overline{x})-\overline{F}_m(\overline{x}_i)\right|+\left|\overline{F}_m(\overline{x}_i)-\overline{F}_n(\overline{x}_i)\right|+\left|\overline{F}_n(\overline{x}_i)-\overline{F}_n(\overline{x}\ )\right| \leq 3\varepsilon.\end{aligned} \tag{8}$$

Then according to the Theorem 7.8 in [7] the sequence $\{\underline{F}_n\},\{\overline{F}_n\}$ are uniformly continuous on $\underline{A},\overline{A}$ and the proof is completed.

Theorem 4.2. Let $\underline{G}_n(\underline{x})$ and $\overline{G}_n(\overline{x})$ are piecewise linear approximations of $\underline{F}(\underline{x}),\overline{F}(\overline{x})$ respectively on $[a,b]$ .as (4). Then:

$$\begin{cases} \underline{G}_n\rightarrow\underline{F} \\ \overline{G}_n\rightarrow\overline{F} \end{cases} unformaly\ \ on[a,b].$$

Proof. The proof is an immediate consequence of Lemma 3.1 and Theorem 3.1.

5 .Extension to nonlinear non-linear non-smooth fuzzy equations

In general it is reasonable to assume that the objective function is a non-smoothness. Therefore we define a kind of generalized differentiation for non-smooth functions in the sense of L1-norm. This kind of differentiation is coinciding with usual differentiation for smooth functions. Therefore the following theorem is represented.

Theorem 5.1. Consider the nonlinear non-smooth function $f : A \to R$ where $A = \prod_{i=1}^{n}[a_i, b_i]$.
Then the optimal solution of the following optimization problem is $f'(x)$.

$$\underset{P(.)}{Minimize} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left| f(x) - (f(s) + (x-s).p(s)) \right| dx_1 \ldots dx_n \tag{9}$$

where $s = (s_1, s_2, \ldots, s_n) \in A$ is an arbitrary point and $P(.) = (P_1(.), P_2(.), \ldots, P_n(.))$ is a vector.

Proof. See [22].

Definition 5.1. Let $f : A \to R$ is a non-smooth function where $A = \prod_{i=1}^{n}[a_i, b_i]$. The global weak differentiation with respect to x in the sense of L1-norm is defined as the $P(.)$ the optimal solution of the minimization problem which is shown in (9).

Now based on Theorem 4.1 and definition 4.1,we proposed extension method for non-linear non-smooth fuzzy functions as following:

Consider $\tilde{F}(x)$ is the nonlinear non-smooth fuzzy function. Based on Theorem 4.1we have non-smooth fuzzy problem as follows:

$$\underset{\tilde{P}(.)}{Minimize} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left| \tilde{F}(x,r) - (\tilde{F}(s,r) + (x-s).\tilde{p}(s,r)) \right| dxds \tag{10}$$

Where : $\tilde{F}(x,r) = \begin{cases} \underline{F}(\underline{x}, \overline{x}, r), \\ \overline{F}(\underline{x}, \overline{x}, r), \end{cases}$, $\tilde{F}(s,r) = \begin{cases} \underline{F}(\underline{s}, \overline{s}, r), \\ \overline{F}(\underline{s}, \overline{s}, r), \end{cases}$, $\tilde{p}(s,r) = \begin{cases} \underline{p}(\underline{s}, \overline{s}, r) \\ \overline{p}(\underline{s}, \overline{s}, r) \end{cases}$ and $r \in [0,1]$.

With use of two step method, non-smooth fuzzy problem convert as follows:

$$\begin{cases} \underset{\underline{P}(.)}{Minimize} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left| \underline{F}(\underline{x}, \overline{x}, r) - (\underline{F}(\underline{s}, \overline{s}, r) + (x-s).\underline{p}(\underline{s}, \overline{s}, r)) \right| d\,\underline{x}\, d\,\underline{s} \\ \underset{\overline{P}(.)}{Minimize} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left| \overline{F}(\underline{x}, \overline{x}, r) - (\overline{F}(\underline{s}, \overline{s}, r) + (x-s).\overline{p}(\underline{s}, \overline{s}, r)) \right| d\overline{x} d\overline{s} \end{cases} \tag{11}$$

And in end, we use of *AVK* method [21] for solving above problems .With suppose $\tilde{x} \in [0, 0.5, 1]$, non-smooth fuzzy minimization problem (9) is formed as:

$$\underset{\tilde{p}(.)}{Minimize} \sum_{i=1}^{n} \left(\int_{i-1/n}^{i/n} \left| \tilde{F}(x,r) - (\tilde{F}(s,r) + (x-s).\tilde{p}(s,r)) \right| dxds \right) \tag{12}$$

Remark: As we know, an approximate value of integral $\int_a^b k(x)dx = (b-a)k(c)$ where c is any point such as: $a \le c \le b.$

So, based two step method, AVK method, applying above remark, and assume c is an ending point in any subinterval and s_i is middle point of $\tilde{x}_i$, (11) is formed as**:**

$$\begin{cases} \underset{\underline{p}(.)}{Minimize} \sum_{r=0}^{1} \sum_{i=1}^{n} \left(\frac{1}{n} \left| \underline{F(\tfrac{i}{n},r)} - (\underline{F(\tfrac{2i-1}{2n},r)} + (\tfrac{-1}{2n}).\underline{p(\tfrac{2i-1}{2n},r)}) \right| \right) \\ \underset{\overline{p}(.)}{Minimize} \sum_{r=0}^{1} \sum_{i=1}^{n} \left(\frac{1}{n} \left| \overline{F(\tfrac{i}{n},r)} - (\overline{F(\tfrac{2i-1}{2n},r)} + (\tfrac{-1}{2n}).\overline{p(\tfrac{2i-1}{2n},r)}) \right| \right) \end{cases} \tag{13}$$

Where : $\tilde{x}_i \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$, $\tilde{s}_i \in \tilde{x}_i = \frac{2i-1}{2n}$ and $\tilde{x}_i - \tilde{s}_i = \frac{-1}{2n}$.As a whole, problems (13) is a NLP problem and we may obtain its solution by many packages such as Lingo,Matlab, Gino, etc.

6. Numerical application

Here we present examples to illustrating the linear parametric approximation method for find positive roots of nonlinear fuzzy equations and nonlinear non-smooth fuzzy equations. Examples 1 and 2 consider from Buckley, Qu [1] and S. Abbasbandy, B. Asady [20].

Example 6.1. Consider the fuzzy nonlinear equation

$$(3,4,5)x^2 + (1,2,3)x = (1,2,3)$$

Without any loss of generality, assume that x is positive and then the parametric form of this equation is as follows:

$$\begin{cases} (3+r)\underline{x}^2 + (1+r)\underline{x} - (1+r) = 0 \\ (5-r)\overline{x}^2 + (3-r)\overline{x} - (3-r) = 0 \end{cases}$$

To obtain initial guess we use above system for $r=0$, $r=1$, therefore:

$$\begin{cases} 4\underline{x}^2(1) + 2\underline{x}(1) = 2 \\ 4\overline{x}^2(1) + 2\overline{x}(1) = 2 \end{cases} \text{and} \begin{cases} 3\underline{x}^2(0) + \underline{x}(0) = 1 \\ 5\overline{x}^2(0) + \overline{x}(0) = 3 \end{cases};$$

we obtain the solution for n=10 and with the Mean squared normalized error(MSE),7.1191e-007. For more details see Figs. 1. Now suppose x is negative, hence $\underline{x}(0) > \overline{x}(0)$, therefore negative root does not exist.

Example 6.2. Consider fuzzy nonlinear equation

$$(1,2,3)x^3 + (2,3,4)x^2 + (3,4,5) = (5,8,13)$$

Without any loss of generality, assume that x is positive and then parametric form of this equation is as follows:

$$\begin{cases} (1+r)\underline{x}^3(r) + (2+r)\underline{x}^2(r) + (3+r) = (5+3r) \\ (3-r)\overline{x}^3(r) + (3-r)\overline{x}^2(r) + (5-r) = (13-5r) \end{cases}$$

Or equality:

$$\begin{cases} (1+r)\underline{x}^3(r) + (2+r)\underline{x}^2(r) - (2+2r) = 0 \\ (3-r)\overline{x}^3(r) + (3-r)\overline{x}^2(r) - (8-4r) = 0 \end{cases}$$

We apply proposed method, for n=10 and show result in Fig. 2 with MSE= 9.6773e-007.

Example 6.3. In this example we consider a nonlinear non-smooth fuzzy function as follows:

$$\tilde{f}(x) = (3,4,5)x^2 + (1,2,3)|x| - (1,2,3)$$

Since objective function is non-smooth fuzzy function. We find the global weak differentiation of $\tilde{f}(x)$ which is the optimal solution of the following optimization problem .we solves this problem based two step method and AVK method for n=10 and with suppose $x \in [0,1]$ as follows:

$$\begin{cases} \underset{\underline{p}(.)}{Minimize} \sum_{r=0}^{1} \left(\int_{-1}^{1} \left| \left[(3+r)\underline{x}^2 + (1+r)|\underline{x}| - (1+r) \right] - \left[(3+r)\underline{s}^2 + (1+r)|\underline{s}| - (1+r) \right] - \underline{p}(s)(\underline{x} - \underline{s}) \right| d\underline{x} \right) \\ \underset{\overline{p}(.)}{Minimize} \sum_{r=0}^{1} \left(\int_{-1}^{1} \left| \left[(5-r)\overline{x}^2 + (3-r)|\overline{x}| - (3-r) \right] - \left[(5-r)\overline{s}^2 + (3-r)|\overline{s}| - (3-r) \right] - \overline{p}(s)(\overline{x} - \overline{s}) \right| d\overline{x} \right) \end{cases}$$

$$\begin{cases} \underset{\underline{p}(.)}{Minimize} \sum_{r=0}^{1} \sum_{i=1}^{n} \left(\frac{1}{n} \left| \left[(3+r)(\tfrac{i}{n})^2 + (1+r)\left|\tfrac{i}{n}\right| - (1+r) \right] - \left[(3+r)(\tfrac{2i-1}{2n})^2 + (1+r)\left|\tfrac{2i-1}{2n}\right| - (1+r) \right] - \underline{p}(s)(\tfrac{-1}{2n}) \right| \right) \\ \underset{\overline{p}(.)}{Minimize} \sum_{r=0}^{1} \sum_{i=1}^{n} \left(\frac{1}{n} \left| \left[(5-r)(\tfrac{i}{n})^2 + (3-r)\left|(\tfrac{i}{n})\right| - (3-r) \right] - \left[(5-r)(\tfrac{2i-1}{2n})^2 + (3-r)\left|(\tfrac{2i-1}{2n})\right| - (3-r) \right] - \overline{p}(s)(\tfrac{-1}{2n}) \right| \right) \end{cases}$$

Last equation is NLP problem and we obtain its solution by Matlab software .The optimal solution is shown in Fig. 3.

Finally we find fuzzy positive root of $\tilde{f}(x)$ based proposed method (piecewise linear approximation) in Fig.4.

Table 5.1 compares approximated and exact solution of last example. Comparison results show the effectiveness of the proposed approach in the presence of fuzzy nonlinear non-smooth functions

Table 5.1-Numerical Results of Example for $x \in [-1,1]$

Nonlinear Non-smooth fuzzy function	Alfa cuts	Mean squared normalized error	Lower Bound		Upper Bound	
			Exact solution	Approximated solution	Exact solution	Approximated solution
$\tilde{f}(x) = (\tilde{4})x^2 + (\tilde{2})\|x\| - (\tilde{2})$	0.0	2.73227459818761e-006	0.434258545912285	0.435064547131094	0.530662386305967	0.530360293406754
	0.1		0.44412415703456	0.444375772558714	0.528341552126349	0.52818178820758
	0.2		0.452934422875685	0.452830255430553	0.525892369611471	0.52588855823833
	0.3		0.460857374128121	0.460540674603175	0.523303729749462	0.523473277978056
	0.4		0.468025837498525	0.468375721500721	0.520563182704725	0.520923045874304
	0.5		0.474546482933995	0.475107991360691	0.517656725750106	0.518227951922027
	0.6		0.480506146710884	0.481295985409024	0.514568548907043	0.515374090827102
	0.7		0.485976360919279	0.487001254955081	0.511280727930352	0.512348971391034
	0.8		0.49101666768945	0.492281879194631	0.507772851213202	0.509135126207242
	0.9		0.495677089411156	0.497181146025878	0.504021563046978	0.505714456061109
	1.0		0.500000000009872	0.501448707909162	0.500000000009872	0.501448707909162

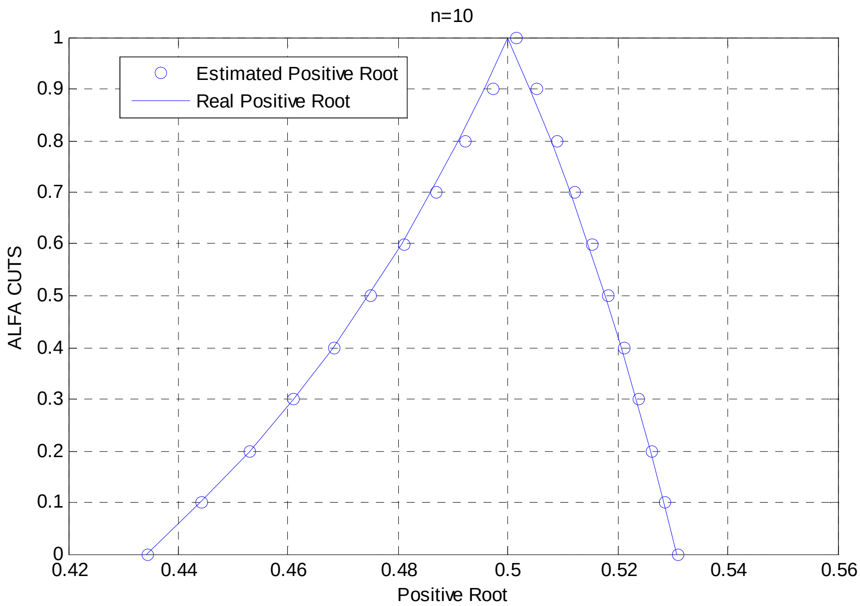

Fig.1. Positive solution of proposed method

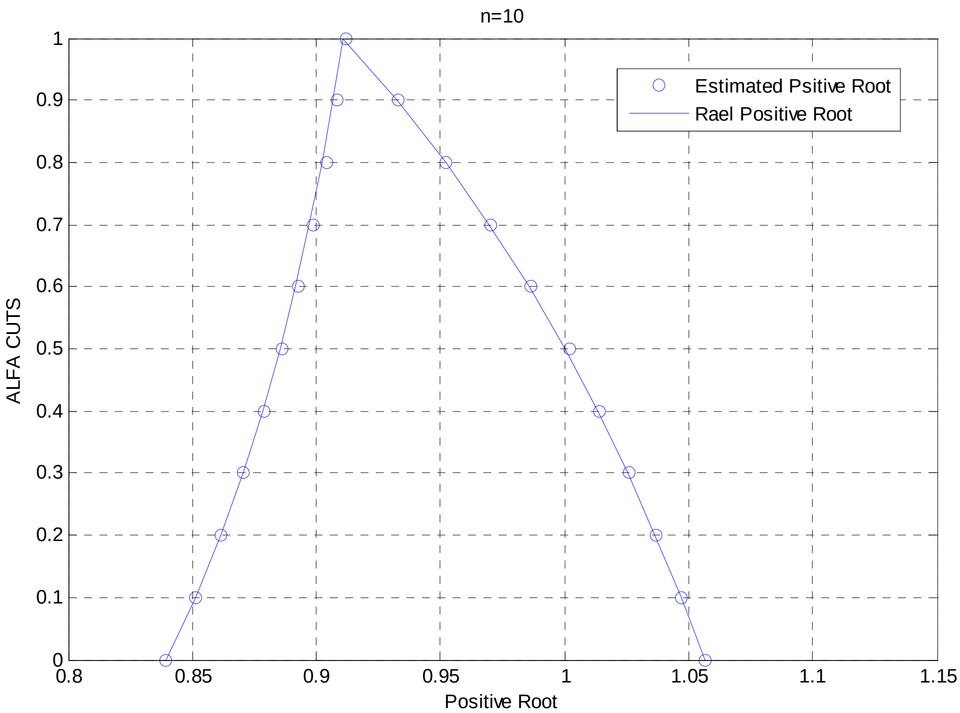

Fig.2. Positive solution and error of proposed method

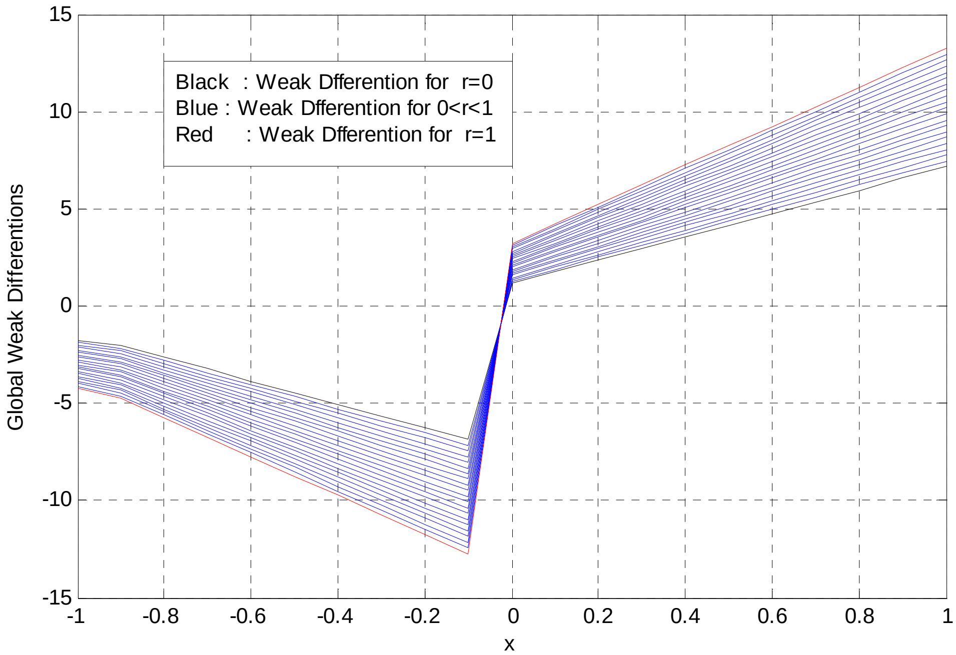

Fig3-Global Weak Differentiation of Nonlinear Non-Smooth Function $\tilde{f}(x) = (3,4,5)x^2 + (1,2,3)|x| - (1,2,3)$

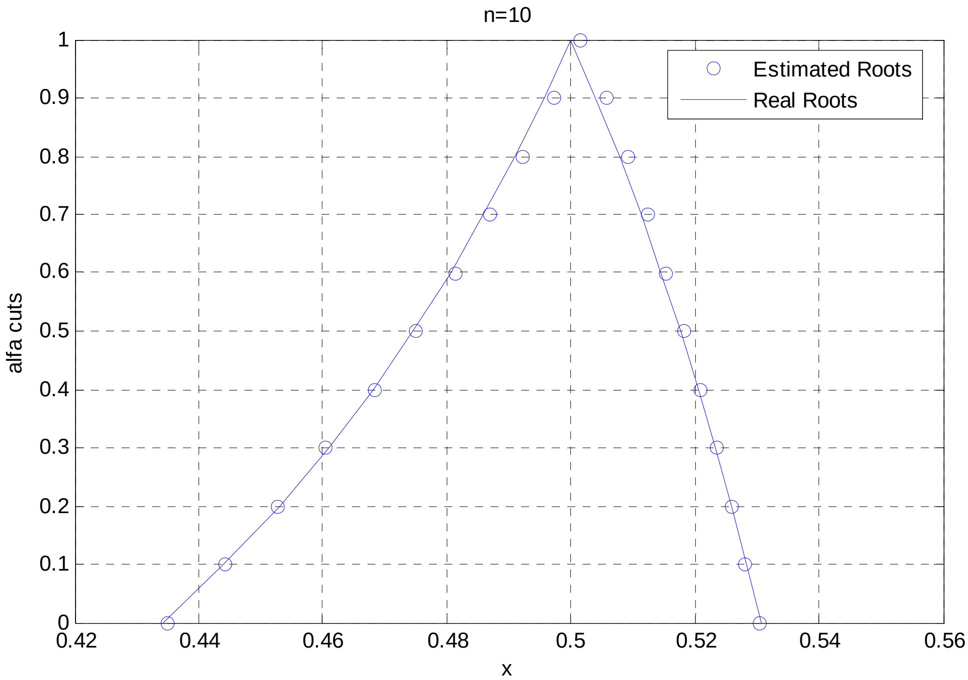

Fig 4. Positive root of non-smooth fuzzy function $\tilde{f}(x)$ **Based Piecewise linear approximation**

CONCLUSION

In this paper, we have suggested numerical solving method for non-linear fuzzy equations instead of standard analytical techniques which are not suitable everywhere. Also the approach can be extended for non-linear non-smooth fuzzy equations by a novel definition of global weak differentiation in the sense of L1 and LP norms. The main advantage of this approach is that we obtained an approximation for the optimum solution of the fuzzy problem with any desirable accuracy. Initially we wrote nonlinear and non-smooth fuzzy equation in parametric form and then solve it by the linear parametric approximation method. Finally, examples were presented to illustrate proposed method.

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