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Velocity, Acceleration and Equations of Motion in the Elliptical Coordinate System

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ABSTRACT

The canonical coordinate systems (rectangular, polar and spherical) are sometimes not the best for studying the trajectories of some forms of motions. For example, motion of objects in an elliptical orbit being described by polar or spherical coordinates may not be accurate. It is due to this that we have derived the position vectors, velocity vectors, acceleration vectors, simple representation of magnitude of the velocity and equations of motion in the elliptical coordinate system. An attempt was also made towards solving the derived equations of motion. The general algorithm for conversion among coordinate systems was also provided.

Keywords: Elliptical coordinate system, Canonical coordinate systems, Equations of motion, Position, Velocity and acceleration

INTRODUCTION

The position, the instantaneous velocity and acceleration of objects are often studied in classical mechanics using rectangular, polar or spherical coordinate system. It is however obvious that these three coordinate system are far too small for the description of all trajectory of particles [1,2]. That is, there are some motion of bodies that this coordinate system cannot fully describe e.g. planetary motion of planets about the sun as the focus, elliptical motion of comets about the sun and many other arbitrary trajectories of bodies that are somehow beyond that which can be described by rectangular, polar, cylindrical and spherical coordinate system. Also, though not a classical problem, the motion of the electron about the nucleus is not also fully circular and neither follows closely any of the mentioned usual coordinate systems. Due to this reason, many types of coordinate system have been formulated to describe the trajectories of moving bodies. Other than those already mentioned, the rest are parabolic cylindrical, oblate spheroidal, elliptical cylindrical, elliptical, paraboloidal, prolate spheroidal, bipolar, toroidal, conical, confocal ellipsoidal, confocal paraboloidal, etc [3,4].

Efforts are being made by researchers in calculating the properties such as position, velocity, acceleration, divergence, gradients, curl, etc in these new coordinate systems. These derivations are necessary and sufficient for expressing all mechanical quantities (linear momentum, kinetic energy, Lagrangian and Hamiltonian) in terms of the new coordinate system coordinates system. The results also pave way for expressing all dynamical laws of motion (Newton's laws, Lagrange's law, Hamiltonian's law, Einstein's Special Relativistic Law of Motion and Schrödinger Law of Quantum Mechanics) entirely in the new coordinate coordinates system [5].

Classically, description of motion of bodies along an arbitrary trajectory is of paramount importance. This is because; many other characteristic of the body can be depicted from its equations of motion. Hence, determining the positions, velocities and accelerations in the various coordinate system interest researchers.

Many researchers have work on this areas and their works have been on the parabolic coordinate system (Omonile, et al.,) elliptical cylindrical coordinate system (Omaghali, et al.), prolate spheroidal coordinate system (Omonile, et al.), rotational spheroidal coordinate system (Omonile, et al,) [1,2,5,6].

Hence, in this work, we shall work on the derivations of position, velocity and acceleration in elliptical coordinate system. This coordinate system was chosen specifically due to its important applicability in description of motion of planets, comets and asteroids about the sun in the solar system. To ease paths for other researchers, we shall also give an algorithm for deriving the position, velocity and accelerations in different coordinate systems. We hope to use the velocity derived in this coordinate system to derive the equations of motions of objects under a central force potential by employing the Euler-Langrange relations. We then make comparisons between the obtained equations of motions in different coordinate systems (Cartesian, polar and elliptical coordinate system).

MATHEMATICAL FORMULATIONS

The most common definition of elliptical coordinate system (u, v) is

$$x = a\cosh u \sin v;\ y = a\sinh u \sin v;\quad [1] \qquad \dots \quad 1$$

Where u is a non-negative real number and $v \in [0,2\pi]$

The elliptical coordinates unit vectors are expressed in terms of the cartesian units vectors as [1]

$$\hat{u} = \frac{\sinh u \cos v\,\hat{i} + \cosh u \sin v\,\hat{j}}{\left(\sinh^2 u + \sin^2 v\right)^{1/2}} \qquad \hat{v} = -\frac{(\cosh u \sin v\,\hat{i} + \sinh u \cos v\,\hat{j})}{\left(\sinh^2 u + \sin^2 v\right)^{1/2}} \qquad \dots \quad 2$$

We can express (2) as:

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} \dfrac{\sinh u \cos v}{\left(\sinh^2 u + \sin^2 v\right)^{1/2}} & \dfrac{\cosh u \sin v}{\left(\sinh^2 u + \sin^2 v\right)^{1/2}} \\ -\dfrac{\cosh u \sin v}{\left(\sinh^2 u + \sin^2 v\right)^{1/2}} & -\dfrac{\sinh u \cos v}{\left(\sinh^2 u + \sin^2 v\right)^{1/2}} \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \end{pmatrix} \qquad \dots\ 3$$

After solving equation (3) for $\hat{i}$ and $\hat{j}$, the values of cartesian unit vectors are:

$$\hat{i} = \frac{\left(\sinh^2 u + \sin^2 v\right)^{1/2}}{\cosh^2 u \sin^2 v - \sinh^2 u \cos^2 v}\left[-\sinh u \cos v\,\hat{u} - \cosh u \sin v\,\hat{v}\right] \qquad \dots \quad 4$$

$$\hat{j} = \frac{\left(\sinh^2 u + \sin^2 v\right)^{1/2}}{\cosh^2 u \sin^2 v - \sinh^2 u \cos^2 v}\left[\cosh u \sin v\,\hat{u} + \sinh u \cos v\,\hat{v}\right] \qquad \dots \quad 5$$

The position vector is given by

$$\vec{r} = x\,\hat{i} + y\,\hat{j} \qquad \dots \quad 6$$

Substituting (1), (4) and (5) in to (6), we get:

$$\vec{r} = \frac{a\left(\sinh^2 u + \sin^2 v\right)^{1/2}}{\cosh^2 u \sin^2 v - \sinh^2 u \cos^2 v}\left[\cosh u \sin u \sinh u\left(\sin^2 v - \cos^2 v\right)\hat{u} - \sin v \cos v\left(\cosh^2 u - \sinh^2 u\right)\hat{v}\right] \qquad (7)$$

The equation above gives the position in the elliptical coordinate system

The velocity is given by

$$\vec{v} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} \qquad \dots \quad 8$$

Before we derived the expression for the velocity, let us state the following substitution we shall use to simplify the results

$$Q(u,v)=\frac{a\left(\sinh^2 u+\sin^2 v\right)^{1/2}}{\cosh^2 u\sin^2 v-\sinh^2 u\cos^2 v}$$

By substituting the first derivatives of equation (1) into (8), we get:

$$\vec{v}=Q(u,v)\left[v_u\hat{u}+v_v\hat{v}\right] \qquad \ldots\ 9$$

Where;

$$v_u=\left[\dot{u}\left(\cosh^2 u\sin^2 v-\sinh^2\sin v\cos v\right)+\dot{v}\text{coshusinhu}\left(\text{sinvcos}\,v-\cos^2 v\right)\right] \qquad \ldots\ 10$$

$$v_v=\left[\dot{u}\text{coshusinhu}\left(\sin v\cos v-\sin^2 v\right)+\dot{v}\left(\sinh^2 u\cos^2 v-\cosh^2\sin v\cos v\right)\right] \qquad \ldots\ 11$$

Equation (9) is the velocity vector equation in the elliptical coordinate system [7,8].

In the application of Euler-Lagrange equation, the magnitude of velocity is however needed and it is given by

$$\left|\vec{v}\right|^2=\left(\dot{x}\right)^2+\left(\dot{y}\right)^2 \qquad \ldots 12$$

After simplification of form obtained from (12), we get;

$$\left|\vec{v}\right|^2=\left(\cosh 2u\sin^2 v\right)\dot{u}^2+\left(\sin^2 v+\sinh^2 u\right)\dot{v}^2+\frac{1}{2}\dot{u}\dot{v}\cosh 2u\sin 2v \qquad \ldots\ 13$$

The acceleration can also be obtained from

$$\vec{a}=\ddot{x}\hat{i}+\ddot{y}\hat{j} \qquad \ldots\ 14$$

By substituting the second derivative of equation (1), (4) and (5) into (14), we get the acceleration as

$$\vec{a}=Q(u,v)\left(a_u\hat{u}-a_v\hat{v}\right) \qquad \ldots 15$$

We shall make use of the following substitution to write the acceleration in a concise form

$$A_1(u,\text{v})=\cosh^2\text{u}\sin^2 v-\sinh^2 u\sin v\cos v \qquad \ldots\ 16$$

$$A_2(u,\text{v})=\sinh u\,\text{coshu}\left(\sin v\cos v-\cos^2 v\right) \qquad \ldots\ 17$$

$$A_3(u,\text{v})=\sinh u\,\text{coshu}\left(\sin^2\text{v}-\sin v\cos v\right) \qquad \ldots\ 18$$

$$A_4(u,\text{v})=\cosh^2 u\cos v\sin v-\sinh^2 u\cos^2 v \qquad \ldots\ 19$$

Hence, the acceleration in the elliptical coordinate system is given by

$$\vec{a}=Q(u,v)\left[\left(A_1\ddot{u}+A_2\ddot{v}+A_3\left(\dot{u}^2-\dot{v}^2\right)+2A_3\dot{u}\dot{v}\right)-\left(A_3\ddot{u}+A_4\ddot{v}+A_1\left(\dot{u}^2-\dot{v}^2\right)+2A_2\dot{u}\dot{v}\right)\right] \qquad \ldots\ 20$$

ALGORITHM FOR CONVERSION AMONG THE VARIOUS COORDINATE SYSTEMS

To help researchers interested in moving from one coordinate system to another, we present general steps for conversion between the various coordinate system. The methods works for any coordinate system provided one has been fed with or has obtained the representation of the interested coordinate system in terms of the Cartesian coordinate system.

1. Express the coordinates of the new coordinate system in terms of the Cartesian coordinate system
2. Express the unit vectors of the new coordinates system in terms of the unit vectors in the Cartesian coordinate system
3. Perform the inversion of equation in step (2). This is done to express the Cartesian unit vectors in terms of the new coordinate system unit vectors. This can be done mostly by solving equation in the step (2) the resulting $n \times n$ matrix

The position in the new coordinate system can therefore be given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Where all the dependent variables have been determined from step (1), (2) and (3).

Hence, velocity, acceleration, the Lagrangian and Hamiltonian in the new coordinate system can be determined once the position is known.

THE EQUATIONS OF MOTION OF OBJECTS IN AN ELLIPTICAL ORBIT

The kinetic energy in the elliptical coordinate system is given by

$$T = \frac{1}{2}m\left[\left(\cosh 2u \sin^2 v\right)\dot{u}^2 + \left(\sin^2 v + \sinh^2 u\right)\dot{v}^2 + \frac{1}{2}\dot{u}\dot{v}\cosh 2u \sin 2v\right] \quad \ldots \quad 21$$

The potential energy is given as shown below if the distance between the sun of mass M and any object of mass m orbiting the elliptical path with the sun as focus is given by:

$$r = x^2 + y^2 \quad \ldots \quad 22$$

Where; $r = x^2 + y^2$ which is equivalent to $r = a\left(\cos^2 v + \sinh^2 u\right)^{1/2}$ in the elliptical coordinate system.

Hence,

$$V = \frac{-GMm}{a}\left[\cos^2 v + \sinh^2 u\right]^{-1/2} \quad \ldots \quad 23$$

Hence, the lagrangian in this coordinate system is given by

$$L = \frac{1}{2}m\left[\left(\cosh 2u \sin^2 v\right)\dot{u}^2 + \left(\sin^2 v + \sinh^2 u\right)\dot{v}^2 + \frac{1}{2}\dot{u}\dot{v}\cosh 2u \sin 2v\right] + \frac{GMm}{a}\left[\cos^2 v + \sinh^2 u\right]^{-1/2} \quad \ldots \quad 24$$

Applying the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) = \frac{\partial L}{\partial u} \quad \ldots \quad 25$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}}\right) = \frac{\partial L}{\partial v} \quad \ldots \quad 26$$

The equations of motion of the particle in the elliptical orbit can be written as:

$$\cosh 2u\left[\ddot{u}\sin^2 v + \ddot{v}\sin 2v\right] = \dot{u}^2 \sin^2 v\left(\sinh u - \sinh 2u\right) + \dot{v}^2 \cosh 2u\left(\sinh u - \cos 2v\right) + \dot{u}\dot{v}\sin 2v\left(\frac{1}{2}\sinh 2u - \cosh 2u\right) - \frac{GM \sinh 2u}{2a\left(\cos^2 v + \sinh^2 u\right)^{3/2}} \quad \ldots \quad 27$$

$$\ddot{v}\left(\sin^2 v+\sinh^2 u\right)+\frac{\ddot{u}}{4}\cosh 2u\sin 2v=\dot{u}^2\sin 2v\left(\frac{1}{2}\cosh 2u-\sinh 2u\right)-\frac{3}{2}\dot{v}^2\sin 2v$$
$$+\dot{u}\dot{v}\left(\cosh 2u\cos 2v-\sinh 2u\right)+\frac{GM\sin 2v}{2a\left(\cos^2 v+\sinh^2 u\right)^{3/2}} \qquad \ldots \quad 28$$

Equations (27) and (28) give the equation of objects moving in a purely elliptical orbit about a larger mass *M*.

Equations (27) and (28) can also be written for the equation of motion of electron of mass m_e about the nucleus of mass M_n as:

$$\cosh 2u\left[\ddot{u}\sin^2 v+\ddot{v}\sin 2v\right]=\dot{u}^2\sin^2 v\left(\sinh u-\sinh 2u\right)+\dot{v}^2\cosh 2u\left(\sinh u-\cos 2v\right)$$
$$+\dot{u}\dot{v}\sin 2v\left(\frac{1}{2}\sinh 2u-\cosh 2u\right)-\frac{\mathrm{keQ}\sinh 2u}{2am_e\left(\cos^2 v+\sinh^2 u\right)^{3/2}} \qquad \ldots \quad 29$$

$$\cosh 2u\left[\ddot{u}\sin^2 v+\ddot{v}\sin 2v\right]=\dot{u}^2\sin^2 v\left(\sinh u-\sinh 2u\right)+\dot{v}^2\cosh 2u\left(\sinh u-\cos 2v\right)$$
$$+\dot{u}\dot{v}\sin 2v\left(\frac{1}{2}\sinh 2u-\cosh 2u\right)+\frac{\mathrm{keQ}\sinh 2v}{2am_e\left(\cos^2 v+\sinh^2 u\right)^{3/2}} \qquad \ldots \quad 30$$

Hence, equations (27) and (28) can be written as the equation of motion for any object of mass m in an elliptical orbit around a larger object of mass M by changing just the potential energy.

Attempted Solution of 27, 28

Equations 27 and 28 are very difficult to solve analytically. The best one can do is to try to simplify or solve them numerically. Equations 27 and 28 are coupled non-linear partial differential equations. We can simplify the coupled equations by making use of the following substitutions 31 and 32. The result being the elimination of time-dependency from equations 27 and 28.

$$\dot{u}=\frac{du}{dv}\dot{v} \qquad \ldots 31$$

$$\ddot{u}=\dot{v}^2\frac{d^2u}{dv^2} \qquad \ldots \quad 32$$

Let there be a constraint such that the rate of change of angular displacement $\dot{v}=k$ is constant.

That is, $\dot{v}=k$ and $\ddot{v}=0$ $\qquad \ldots \quad 33$

Substituting 31, 32 and 33 into 27, we get:

$$\frac{d^2u}{dv^2}-\left(2-4\tanh 2u\right)\left(\frac{du}{dv}\right)^2-4\left(\cot 2v-\frac{\tanh 2u}{\sin 2v}\right)\frac{du}{dv}+\frac{6}{\cosh 2u}$$
$$-\frac{2GM}{ak^2\cosh 2u}\left(\cos^2 v+\sinh^2 u\right)^{-3/2}=0 \qquad \ldots \quad 34$$

Substituting 31, 32 and 33 into 28, we get:

$$\frac{d^2u}{dv^2}-\left(\frac{\sinh u}{\cosh 2u}-\tanh 2u\right)\left(\frac{du}{dv}\right)^2-\left(\cot v-\cos ec^2 v\right)\frac{du}{dv}-\left(\frac{\sinh u}{\sin^2 v}-\frac{\cos 2v}{\sin^2 v}\right)$$
$$+\frac{GM}{2ak^2}\frac{\tanh 2u}{\sin^2 v}\left(\cos^2 v+\sinh^2 u\right)^{-3/2}=0 \quad \dots \quad 35$$

Equations 34 minus 35 give

$$\left(3\tanh 2u+\frac{\sinh u}{\cosh 2u}-2\right)\left(\frac{du}{dv}\right)^2+\left(\cot v-4\cot 2v+\frac{4\tanh u}{\sin 2v}-\frac{1}{\sin^2 v}\right)\frac{du}{dv}$$
$$+\left(\frac{6}{\cosh 2u}+\frac{\sinh u}{\sin^2 v}-\frac{\cos 2v}{\sin^2 v}\right)+\frac{GM}{2ak^2}\left(\cos^2 v+\sinh^2 u\right)^{-3/2}\left(\frac{-4}{\cosh 2u}-\frac{\tanh 2u}{\sin^2 v}\right)=0 \quad \dots \quad 36$$

Equation 36 is a non-linear ordinary differential equation which ordinarily is very easy to solve analytically. However, it has lots of transcendental and hyperbolic coefficients that will make its solution almost impossible either numerically or analytically.

This difficulty in solving the equation of motion obtained probably explains why the literature rather choose the polar coordinate for the representation of objects naturally in the elliptical orbit, though using the polar coordinate for the description may not be as accurate using the elliptical coordinate system.

Since we have not been able to solve the resulting equations of motion 27 and 28, it is important to compare with the equations of motion in some other known coordinate systems.

The equations of motion of object of mass m under a central force potential in the Cartesian coordinate system is given by [11] as:

$$\frac{d^2x}{dt^2}=-GM\frac{x}{\left(x^2+y^2\right)^{3/2}} \quad \dots \quad 37$$

$$\frac{d^2y}{dt^2}=-GM\frac{y}{\left(x^2+y^2\right)^{3/2}}\ [9] \quad \dots \quad 38$$

By keen inspection of 27, 28 and 37, 38, one notices some similarities but the most important of the similarities is the fact that the equations in both coordinate system cannot be solved analytically because they are both non-linear. However, 37 and 38 can be linearized by conversion to polar coordinates using the transformation: $x=r\cos\theta$ and $y=r\sin\theta$ and obtain:

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=\frac{-GM}{r^2} \quad \dots \quad 39$$

$$r^2\frac{d^2r}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}=0\ [9] \quad \dots \quad 40$$

Equations 39 and 40 describe the trajectory of a planet in polar coordinates. However, these equations are still non-linear. To linearize further, we employ the conservation of angular momentum for bodies under the influence of central force. That is, $mr^2\dot{\theta}=L$

$$\frac{d^2r}{dt^2}-\frac{L^2}{m^2}\frac{1}{r^3}=\frac{-GM}{r^3} \quad \dots \quad 41$$

$$u = \frac{1}{r} \quad [11] \qquad \ldots \qquad 42$$

Equations 41 and 42 are the popular orbit equation that we are familiar with. 41 is still nonlinear while 42 is the representation of the angular momentum conservation. So, we make use of substitution $u = \frac{1}{r}$ to further linearize equation 41 and we get:

$$\frac{d^2u}{d\theta^2} + u = GM\left(\frac{L}{m}\right)^2 \qquad \ldots 43$$

Equation 43 is now a linear differential equation which can be solved analytically.

However, in our situation of the elliptical coordinated system, it takes more than just conservation of angular momentum to linearize the differential equations (27 and 28). While the conservation theorem linearizes the equations with respect to "v", the equations still remains non-linear with respect to "*u*". Another alternative to linearizing 27 and 28 is to transform the coordinates to polar representation where momentum conservation is enough for linearization. Otherwise, the differential equations (27 and 28) remain non-linear. This is also the case in the Cartesian coordinate system as depicted in equations 37 and 38 where angular momentum is not also enough for linearization. Hence, 37 and 38 cannot also be solved analytically. Only numerical solution is possible for 37 and 38.

Provided, non-linearity is the only hindrance to solving the differential equations (27 and 28), we would have employed a numerical solution straight forwardly. However, the coefficients of the differentials are combinations of hyperbolic and transcendental functions. These make the differential equations highly difficult to solve, even numerically.

CONCLUSION

So far in this paper, we have highlighted the algorithms for converting the parameters of a coordinate system to another. With this, we derived the vectors for the position, velocity and acceleration in the elliptical coordinate system. We also derived a simple way of expressing the magnitude of the velocity in the coordinate system [10,11]. Overall, we derived the equations of motion of objects in the elliptical orbit for objects under an inverse-square, central, attractive potential. However, the resulting equation remains non-linear despite some attempts to remove the non-linearity. Moreover, the presence of hyperbolic and transcendental coefficients introduce more difficulty in solving the equations of motion in the elliptical coordinate system. Hence, though the natural orbits of planets is elliptical, the description of the trajectories of the planets is possible only in polar coordinate system, at least analytically.

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